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DUAL POLYNOMIALS AND COMPLEX DUAL NUMBERS FOR ANALYSIS OF SPATIAL MECHANISMS

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ABSTRACT

Complex dual numbers $\breve{w}_1 = x_1 + iy_1 + \varepsilon u_1 + i\varepsilon v_1$ which form a commutative ring are for the first time introduced in this paper. Arithmetic operations and functions of complex dual numbers are defined. Complex dual numbers are used to solve dual polynomial equations. It is shown that the singularities of a dual input-output displacement polynomial equation of a mechanism correspond to its singularity positions. This new method of identifying singularities provides clear physical insight into the geometry of the singular configurations of a mechanism, which is illustrated through analysis of special configurations of the RCCC spatial mechanism. Numerical solutions for dual polynomial equations and complex dual numbers are conveniently implemented in the C$^{\mathrm{H}}$ language environmentfor analysis of the RCCC spatial mechanism. Like the dual number, the complex dual number is a useful mathematical tool for analytical and numerical treatment of spatial mechanisms.

INTRODUCTION

A variety of methods can be used for analysis and design of spatial mechanisms. The popular displacement operators or coordinate transformation forms include 4x4 Denavit-Hartenberg matrices (Denavit and Hartenberg, 1955), 4x4 screw matrices (Suh and Radcliffe, 1978), 3x3 dual matrices (Soni and Harrisberger, 1968; Yang, 1969; Pennock and Yang, 1985; McCarthy, 1990), 6x6 line coordinate transformation matrices (Yuan, 1970, Sugimoto, et al., 1982), 2x2 dual matrices (Denavit, 1958), dual quaternions (Yang and Freudenstein, 1964), etc. A 3x3 dual transformation matrix is orthogonal and has a format similar to a 3x3 pure rotational matrix with real elements. Because of their compact nature, the 3x3 dual matrices are widely used for analytical treatment of spatial mechanisms. However, formulas with dual numbers are difficult to handle in traditional programming languages such as FORTRAN and C. Although succinct dual formulations for mechanism analysis and design can be derived, their numerical computation tends to become cumbersome. For numerical implementation, dual equations have to be reformulated as complicated formulas with only real numbers. The process of rewriting equations with only real numbers is complicated and error-prone (Yang, 1969).

To simplify the task of analysis and design of spatial mechanisms formulated with dual numbers, we have implemented dual data type in the C$^{\mathrm{H}}$ language environment (Cheng, 1994; 1995). Formulations under the C$^{\mathrm{H}}$ programming paradigm is significantly different from those under other programming paradigms. We have demonstrated the simplicity of analysis of spatial mechanisms in both closed-form and iterative formulations under the C$^{\mathrm{H}}$ programming paradigm (Cheng and Thompson, 1995; 1996). In this paper, some analytical and numerical aspects of dual numbers related to analysis of spatial mechanisms will be presented. Because dual numbers form a commutative ring, a dual polynomial possesses some special properties in comparison with a real polynomial. Of particular interest is that singularities of a dual input-output displacement polynomial of a mechanism correspond to its singularity configurations. We will introduce a new number system called complex dual number in this paper. The need of this new number system stems from the numerical solution of dual polynomial equations. A numerical procedure of solving general dual polynomial equations based on complex dual numbers using the C[H] language environment will be presented in the paper.

Researchers working in the area of mechanism and robotics have given much attention to kinematic singularities for the purpose of control, inverse kinematics, design, etc. There are a variety of ways to identify the singularity configurations of a mechanism. The singular positions of a mechanism can be identified if the determinant of the Jacobian is zero (Lipkin and Duffy, 1985), or if the Jacobian is rank deficient (Litvin, et al, 1986; Wang and Waldron, 1987). The singularities of linkages can be detected by the input-output function and its derivative (Ranjbaran, et al., 1992). The singularity can also be determined by monitoring the Gramian of the screws which represent the instantaneous motion of the joints using the matrix of Plücker coordinates (Sugimoto, et al., 1982). In general, identification of kinematic singularities through matrices does not provide physical insight into the nature of singular configurations. In this paper, a new method of identification of singularity of mechanisms based on the dual polynomial will be presented. The method provides clear physical insight into the geometry of the singular configurations of a mechanism. This novel approach is illustrated through the analysis of the special configurations of the RCCC spatial mechanism.

DUAL POLYNOMIALS

DEFINITION OF DUAL POLYNOMIALS

If the roots of a dual polynomial are dual numbers, an nth order dual polynomial can be defined as

$$\widehat{d}_n\widehat{x}^n + \widehat{d}_{n-1}\widehat{x}^{n-1} + ... + \widehat{d}_1\widehat{x}^1 + \widehat{d}_0 = 0 \tag{1}$$

where $\widehat{d}_i \in \mathcal{D} = \{(\acute{d}_i, \grave{d}_i)|\acute{d}_i, \grave{d}_i \in \mathcal{R}\}$ and $\widehat{x} \in \mathcal{D} = \{(\acute{x}, \grave{x})|\acute{x}, \grave{x} \in \mathcal{R}\}$ are dual numbers with $\mathcal{R}$ being a set of real numbers. $\acute{d}_i$ and $\grave{d}_i$ are real and dual parts of $\widehat{d}_i$, and $\acute{x}$ and $\grave{x}$ are the real and dual parts of $\widehat{x}$, respectively. Equation (1) can be written as

$$\sum_{i=0}^{n} \widehat{d}_i\widehat{x}^i = 0. \tag{2}$$

Operations of addition and multiplication of real polynomials can be applied to dual polynomials as well. For example, given two dual polynomials

$$f(\widehat{x}) = \sum_{i=0}^{n} \widehat{d}_i\widehat{x}^i \tag{3}$$

and

$$g(\widehat{x}) = \sum_{i=0}^{n} \widehat{c}_i\widehat{x}^i, \tag{4}$$

addition and multiplication of two dual polynomials can be defined as

$$f(\widehat{x}) + g(\widehat{x}) = \sum_{i=0}^{n} (\widehat{d}_i + \widehat{c}_i)\widehat{x}^i \tag{5}$$

and

$$f(\widehat{x})g(\widehat{x}) = \sum_{i=0}^{n} \left(\sum_{j=0}^{i} \widehat{d}_j\widehat{c}_{i-j}\right)\widehat{x}^i, \tag{6}$$

respectively.

SINGULARITIES IN A DUAL POLYNOMIAL

Some special features of dual polynomials will be presented in this section. First, we will demonstrate how $\acute{x}$ relates to $\grave{x}$ in a second-order polynomial. A general second-order polynomial can be written as

$$\widehat{d}_2\widehat{x}^2 + \widehat{d}_1\widehat{x} + \widehat{d}_0 = 0. \tag{7}$$

Substituting $\widehat{d}_i = \acute{d}_i + \varepsilon\grave{d}_i$ and $\widehat{x} = \acute{x} + \varepsilon\grave{x}$ with $\varepsilon^2 = 0$ into equation (7), we get

$$(\acute{d}_2+\varepsilon\grave{d}_2)(\acute{x}^2+2\varepsilon\acute{x}\grave{x})+(\acute{d}_1+\varepsilon\grave{d}_1)(\acute{x}+\varepsilon\grave{x})+(\acute{d}_0+\varepsilon\grave{d}_0) = 0. \tag{8}$$

Separating the real and dual part of equation (8), we can obtain the following two equations:

$$\acute{d}_2\acute{x}^2 + \acute{d}_1\acute{x} + \acute{d}_0 = 0 \tag{9}$$

$$(2\acute{d}_2\acute{x} + \acute{d}_1)\grave{x} + \grave{d}_2\acute{x}^2 + \grave{d}_1\acute{x} + \grave{d}_0 = 0 \tag{10}$$

From equation (9), we can solve for the real part $\acute{x}$ of $\widehat{x}$ of solutions to equation (7). The dual part $\grave{x}$ of $\widehat{x}$ then can be obtained from

$$\grave{x} = -\frac{\grave{d}_2\acute{x}^2 + \grave{d}_1\acute{x} + \grave{d}_0}{2\acute{d}_2\acute{x} + \acute{d}_1}. \tag{11}$$

Theorem 1: *For a second-order dual polynomial equation (7), if there are two different values for the real part $\acute{x}$ in solving equation (9), we can obtain two different values for the dual part $\grave{x}$ from equation (11). However, if two solutions for $\acute{x}$ of equation (9) are identical, the dual polynomial equation (7) is singular. The solutions of a singular dual polynomial equation depend on the value of the determinant*

$$\Delta = \grave{d}_2\acute{x}^2 + \grave{d}_1\acute{x} + \grave{d}_0 \tag{12}$$

If the determinant (12) is zero, there are infinite solutions for $\grave{x}$ of $\widehat{x}$ for equation (7). But, if the determinant (12) is not zero, the solution for the dual part $\grave{x}$ of $\widehat{x}$ is infinity. If $\acute{x}_1$ and $\acute{x}_2$ are approximation of two identical roots, the dual part $\grave{x}$ approaches positive and negative infinity.

This theorem can be proved as follows. Let two solutions to equation (9) be $\acute{x}_1$ and $\acute{x}_2$; then

$$\acute{d}_2\acute{x}_1^2 + \acute{d}_1\acute{x}_1 + \acute{d}_0 = 0 \tag{13}$$

$$\acute{d}_2\acute{x}_2^2 + \acute{d}_1\acute{x}_2 + \acute{d}_0 = 0. \tag{14}$$

Subtracting equation (13) from equation (14) gives

$$\acute{d}_2(\acute{x}_2^2 - \acute{x}_1^2) + \acute{d}_1(\acute{x}_2 - \acute{x}_1) = 0 \tag{15}$$

or

$$(\acute{x}_2 - \acute{x}_1)[\acute{d}_2(\acute{x}_2 + \acute{x}_1) + \acute{d}_1] = 0. \tag{16}$$

Since $\acute{x}_2$ and $\acute{x}_1$ are not always equal, then

$$\acute{d}_2(\acute{x}_2 + \acute{x}_1) + \acute{d}_1 = 0. \tag{17}$$

In the case when $\acute{x}_1$ and $\acute{x}_2$ are identical, equation (17) becomes

$$2\acute{d}_2\acute{x} + \acute{d}_1 = 0 \tag{18}$$

Then, equation (10) becomes

$$\grave{d}_2\acute{x}^2 + \grave{d}_1\acute{x} + \grave{d}_0 = 0. \tag{19}$$

If equation (19) is satisfied, $\grave{x}$ in equation (10) can have any value because of equation (18). In the case when the two roots are nearly identical, say $\acute{x} = x \pm \delta$, where x is the exact solution and δ is a small perturbation. Equation (11) becomes

$$\grave{x} = -\frac{\grave{d}_2x^2 + \grave{d}_1x + \grave{d}_0}{\pm 2\acute{d}_2\delta}. \tag{20}$$

after ignoring terms of higher order. As δ approaches zero, $\grave{x}$ becomes $\pm\,\infty$.

For example, the dual polynomial equation

$$(\widehat{x} + 3 + \varepsilon 4)(\widehat{x} + 5 + \varepsilon 6) = 0 \tag{21}$$

or

$$\widehat{x}^2 + (8 + \varepsilon 10)\widehat{x} + (15 + \varepsilon 38) = 0 \tag{22}$$

has unique solutions. The dual polynomial equation

$$(\widehat{x} + 9 + \varepsilon 2)(\widehat{x} + 9 + \varepsilon 8) = 0 \tag{23}$$

or

$$\widehat{x}^2 + (18 + \varepsilon 10)\widehat{x} + (81 + \varepsilon 90) = 0 \tag{24}$$

has two identical real parts for its solutions. Hence, the dual number $\widehat{x} = -9 - \varepsilon y$ is a solution to equation (24), where y can be any finite number. Using the dual quadratic formula

$$\widehat{x} = \frac{-\widehat{d}_1 \pm \sqrt{\widehat{d}_1^2 - 4\widehat{d}_2\widehat{d}_0}}{2\widehat{d}_2}. \tag{25}$$

for equation (7), the roots of equation (24) are $\widehat{x} = -9 - \varepsilon 5$. The solution for the dual polynomial equation

$$\widehat{x}^2 + (18 + \varepsilon 10)\widehat{x} + (81 + \varepsilon\grave{d}_0) = 0 \tag{26}$$

is

$$\widehat{x} = \frac{-(18 + \varepsilon 10) \pm \sqrt{\varepsilon(360 - 4\grave{d}_0)}}{2}. \tag{27}$$

If $\grave{d}_0 \neq 90$, the two solutions are $\widehat{x} = -9 + \varepsilon\infty$ and $\widehat{x} = -9 - \varepsilon\infty$. Note that the square root function of a dual number is defined as $\sqrt{\widehat{d}} = \sqrt{\acute{d}} + \varepsilon\grave{d}/\sqrt{\acute{d}}$. The square root of a pure dual number is a pure dual infinity.

An nth order dual polynomial equation can be written as

$$\widehat{d}_n\widehat{x}^n + \widehat{d}_{n-1}\widehat{x}^{n-1} + \widehat{d}_{n-2}\widehat{x}^{n-2} + \cdots + \widehat{d}_1\widehat{x} + \widehat{d}_0 = 0. \tag{28}$$

or

$$\begin{aligned}(\acute{d}_n + \varepsilon\grave{d}_n)(\acute{x} + \varepsilon\grave{x})^n + (\acute{d}_{n-1} +{}& \\ \varepsilon\grave{d}_{n-1})(\acute{x} + \varepsilon\grave{x})^{n-1} + (\acute{d}_{n-2} +{}& \\ \varepsilon\grave{d}_{n-2})(\acute{x} + \varepsilon\grave{x})^{n-2} + \cdots +{}& \\ (\acute{d}_1 + \varepsilon\grave{d}_1)(\acute{x} + \varepsilon\grave{x}) + (\acute{d}_0 + \varepsilon\grave{d}_0) = 0.&\end{aligned} \tag{29}$$

Note that expanding the dual number $(\acute{x} + \varepsilon\grave{x})^n$ results in $\acute{x}^n + \varepsilon n\acute{x}^{n-1}\grave{x}$. The equation can be separated into its real and dual parts. The real part is

$$\acute{d}_n\acute{x}^n + \acute{d}_{n-1}\acute{x}^{n-1} + \acute{d}_{n-2}\acute{x}^{n-2} + \cdots + \acute{d}_1\acute{x} + \acute{d}_0 = 0 \tag{30}$$

and the dual part is

$$\begin{aligned}[n\acute{d}_n\acute{x}^{n-1} + (n-1)\acute{d}_{n-1}\acute{x}^{n-2} +{}& \\ (n-2)\acute{d}_{n-2}\acute{x}^{n-3} + \cdots + \acute{d}_1]\grave{x} +{}& \\ \grave{d}_n\acute{x}^n + \grave{d}_{n-1}\acute{x}^{n-1} + \grave{d}_{n-2}\acute{x}^{n-2} + \cdots + \grave{d}_1\acute{x} + \grave{d}_0 = 0&\end{aligned} \tag{31}$$

or

$$\grave{x} = -\frac{\grave{d}_n\acute{x}^n + \grave{d}_{n-1}\acute{x}^{n-1} + \grave{d}_{n-2}\acute{x}^{n-2} + \cdots + \grave{d}_1\acute{x} + \grave{d}_0}{n\acute{d}_n\acute{x}^{n-1} + (n-1)\acute{d}_{n-1}\acute{x}^{n-2} + (n-2)\acute{d}_{n-2}\acute{x}^{n-3} + \cdots + \acute{d}_1} \tag{32}$$

Theorem 1 can be easily extended to the solution of an nth order dual polynomial equation as follows.

Theorem 2: *For an nth order dual polynomial, equation (28), if there are two identical values for the real part $\acute{x}$ in solving equation (30), the dual polynomial equation (28) is singular. The solutions of a singular dual polynomial equation depend on the value of the determinant*

$$\Delta = \grave{d}_n \acute{x}^n + \grave{d}_{n-1} \acute{x}^{n-1} + \grave{d}_{n-2} \acute{x}^{n-2} + \cdots + \grave{d}_1 \acute{x} + \grave{d}_0. \quad (33)$$

If the determinant (33) is zero, there are infinite solutions for the dual part $\grave{x}$ of $\widehat{x}$ corresponding to the identical real component. But, if the determinant is not zero, the solution for $\grave{x}$ corresponding to the identical real component is infinity.

For example, the dual polynomial equation formed by the roots $\widehat{x}_1 = (3+\varepsilon)$, $\widehat{x}_2 = (3+\varepsilon 5)$, and $\widehat{x}_3 = 9+\varepsilon 2$ is

$$\widehat{x}^3 - (15+\varepsilon 8)\widehat{x}^2 + (63+\varepsilon 84)\widehat{x} - (81+\varepsilon 180) = 0. \quad (34)$$

For this third order polynomial with identical real roots $\acute{x} = \acute{x}_1 = \acute{x}_2 = 3$, with $\grave{d}_3 = 0$, $\grave{d}_2 = -8$, $\grave{d}_1 = 84$, $\grave{d}_0 = -180$, and $\acute{x} = 3$, the determinant $\grave{d}_3 \acute{x}^3 + \grave{d}_2 \acute{x}^2 + \grave{d}_1 \acute{x} + \grave{d}_0$ equals zero. Therefore, the dual part $\grave{x}$ of $\widehat{x} = 3 + \varepsilon \grave{x}$ can be any value.

NUMERICAL SOLUTIONS TO A DUAL POLYNOMIAL

Two numerical methods for solving dual polynomial equations have been developed. A method based on the idea that the eigenvalues of a polynomial's companion matrix are the roots of the polynomial is given in (Thompson, 1996). Analog to the derivation for solutions of a real polynomial in (Press, 1988), a modified Laguerre formulation using dual numbers to solve a dual polynomial equation is described in this section.

Assume that the dual polynomial $\widehat{P}_n$ is formed by multiplying n dual roots together.

$$\widehat{P}_n(\widehat{x}) = (\widehat{x} - \widehat{x}_1)(\widehat{x} - \widehat{x}_2)\ldots(\widehat{x} - \widehat{x}_n) \quad (35)$$

Taking the natural logarithm of equation (35) gives

$$\ln(\widehat{P}_n(\widehat{x})) = \ln(\widehat{x} - \widehat{x}_1) + \ln(\widehat{x} - \widehat{x}_2) + \ldots + \ln(\widehat{x} - \widehat{x}_n). \quad (36)$$

The first derivative of equation (36) is

$$\frac{d\ln(\widehat{P}_n(\widehat{x}))}{d\widehat{x}} = \frac{1}{\widehat{x} - \widehat{x}_1} + \frac{1}{\widehat{x} - \widehat{x}_2} + \ldots + \frac{1}{\widehat{x} - \widehat{x}_n} = \frac{\widehat{P}'_n}{\widehat{P}_n} = \widehat{G}. \quad (37)$$

let

$$\widehat{a} = \widehat{x} - \widehat{x}_1 \quad (38)$$

and assume

$$\widehat{b} = \widehat{x} - \widehat{x}_i, \text{ for } i = 2, 3, \ldots, n. \quad (39)$$

Substituting equations (38) and (39) into equation (37), we get

$$\frac{d\ln(\widehat{P}_n(\widehat{x}))}{d\widehat{x}} = \frac{1}{\widehat{x} - \widehat{x}_1} + \frac{1}{\widehat{b}} + \ldots + \frac{1}{\widehat{b}} = \frac{1}{\widehat{a}} + \frac{n-1}{\widehat{b}} = \widehat{G}, \quad (40)$$

where $\widehat{G}$ is an approximation of the first derivative of the polynomial. To obtain an approximation of the second derivative of the polynomial, take the second derivative of equation (36). This gives

$$\begin{aligned} \frac{d^2\ln(\widehat{P}_n(\widehat{x}))}{d\widehat{x}^2} &= \frac{1}{(\widehat{x} - \widehat{x}_1)^2} + \frac{1}{(\widehat{x} - \widehat{x}_2)^2} + \ldots + \frac{1}{(\widehat{x} - \widehat{x}_n)^2} \\ &= \frac{\widehat{P}'_n}{\widehat{P}_n} - \frac{\widehat{P}''_n}{\widehat{P}_n} = \widehat{H}. \end{aligned} \quad (41)$$

Substituting equations (38) and (39) into equation (41) results in

$$\begin{aligned} \frac{d^2\ln(\widehat{P}_n(\widehat{x}))}{d\widehat{x}^2} &= \frac{1}{(\widehat{x} - \widehat{x}_1)^2} + \frac{1}{\widehat{b}^2} + \ldots + \frac{1}{\widehat{b}^2} \\ &= \frac{1}{\widehat{a}^2} + \frac{n-1}{\widehat{b}^2} \\ &= \widehat{H}. \end{aligned} \quad (42)$$

Combining equations (40) and (42) yields the solution for $\widehat{a}$

$$\widehat{a} = \frac{n}{\widehat{G} \pm \sqrt{(n-1)(n\widehat{H} - \widehat{G}^2)}}, \quad (43)$$

where $\widehat{a}$ is the deviation from our current approximated root $\widehat{x}$, n is the power of the polynomial, $\widehat{H}$ is an approximation of the first derivative of the polynomial, $\widehat{G}$ is an approximation of the second derivative, and the sign should be taken to yield the largest magnitude for the denominator. The

Table 1. COMPLEX DUAL OPERATIONS WITH $\check{w}_1 = (z_{11},\, z_{12}) = x_1 + iy_1 + \varepsilon u_1 + i\varepsilon v_1$ AND $\check{w}_2 = (z_{21},\, z_{22}) = x_2 + iy_2 + \varepsilon u_2 + i\varepsilon v_2$

Operation	Definition
$-\check{w}_1$	$-z_{11} - \varepsilon z_{12}$
$\check{w}_1 + \check{w}_2$	$(z_{11} + z_{21}) + \varepsilon(z_{12} + z_{22})$
$\check{w}_1 - \check{w}_2$	$(z_{11} - z_{21}) + \varepsilon(z_{12} - z_{22})$
$\check{w}_1 * \check{w}_2$	$z_{11} * z_{21} + \varepsilon(z_{12} * z_{21} + z_{11} * z_{22})$
$\check{w}_1 / \check{w}_2$	$\dfrac{z_{11}}{z_{21}} + \varepsilon \dfrac{z_{12} * z_{21} - z_{11} * z_{22}}{z_{21}^2}$

next iteration to determine the new value of $\widehat{x}$ is calculated by $\widehat{x} = \widehat{x} - \widehat{a}$. As $\widehat{a}$ gets smaller, $\widehat{x}$ gets closer to the correct root. $\widehat{H}$ and $\widehat{G}$ are approximated by a simple algorithm that incorporates $\widehat{x}$ and the polynomial coefficients.

After one root is found, deflation is applied to the polynomial to obtain the coefficients that describe the remaining polynomial. The above process is then repeated to the remaining polynomial to find each successive root. However, this numerical process will fail if the roots are restricted to dual numbers only. Any time a negative dual number occurs under the radical in equation (43), the value will become invalid. The deflation process then cannot be continued. Therefore, the *complex dual number* is introduced to solve dual polynomial equations numerically. In our numerical implementation, complex dual numbers are used to replace dual numbers in equations (37) to (43). Details about complex dual numbers are described in the next section.

COMPLEX DUAL NUMBERS

A *complex dual* number is defined as an ordered pair of complex numbers

$$\check{w} = (z_1, z_2) = (x + iy) + \varepsilon(u + iv), \tag{44}$$

where $\check{w} \in \mathcal{W} = \{(z_1, z_2) | z_1, z_2 \in \mathcal{C}\}$ or W $= \{(x, y, u, v) | x, y, u, v \in \mathcal{R}\}$. A *dual complex* number is defined as an ordered pair of dual numbers,

$$\check{w} = (\widehat{d}_1, \widehat{d}_2) = (x + \varepsilon u) + i(y + \varepsilon v), \tag{45}$$

where $\check{w} \in \mathcal{W} = \{(\widehat{d}_1, \widehat{d}_2) | \widehat{d}_1, \widehat{d}_2 \in \mathcal{D}\}$ or $\mathcal{W} = \{(x, y, u, v) | x, y, u, v \in \mathcal{R}\}$. The conjugate of a dual complex number is defined as $\overline{\check{w}} = (x + \varepsilon u) - i(y + \varepsilon v)$. A *complex dual* number or dual complex number can be written in terms of its base elements $(1,\, i,\, \varepsilon,\, i\varepsilon)$ as

$$\check{w} = x + iy + \varepsilon u + i\varepsilon v. \tag{46}$$

where x is called the real part, y the imaginary part, u the dual part, and v the imaginary dual part. Both complex dual numbers and dual complex numbers are identical. For consistency, we will call $\check{w}$ a complex dual number in the remaining part of this paper. The base elements $(1,\, i,\, \varepsilon,\, i\varepsilon)$ of complex dual numbers satisfy the following properties:

$$1 * i = i \tag{47}$$
$$1 * \varepsilon = \varepsilon \tag{48}$$
$$i * i = -1 \tag{49}$$
$$\varepsilon * \varepsilon = 0 \tag{50}$$
$$i * \varepsilon = i\varepsilon \tag{51}$$
$$i * i\varepsilon = -\varepsilon \tag{52}$$
$$\varepsilon * i\varepsilon = 0, \tag{53}$$

where ε designates the *pure dual* unit, i designates the imaginary unit, and $i\varepsilon$ designates the imaginary dual unit. If $x \neq 0$, $u \neq 0$, and $y = v = 0$, $x + \varepsilon u$ is called a *proper dual* number. If all values of x, y, u, and v are not zero, $x + iy + \varepsilon u + i\varepsilon v$ is called a *proper complex dual* number.

The arithmetic operations and functions of complex dual numbers are defined in Tables 1 and 2, respectively. Complex operations and functions in these tables are defined in (Cheng, 1993). Essentially, Tables 1-2 are obtained by replacing the ordered pair of real numbers of a dual number in dual operations and functions in (Cheng, 1994) with the ordered pair of complex numbers. Like dual numbers, complex dual numbers form a commutative ring. Table 3 summarizes some properties of dual complex numbers. Application examples of complex dual numbers are given in the next section.

Note that the absolute value of a dual complex number in Table 2 is defined such that properties 4, 8, and 10 in Table 3 hold true. This definition is different from the original definition of the absolute value of a dual number in (Cheng, 1993). According to this new definition, the absolute value of a dual number is a dual number rather than a real number. To make up for the change in returned values, the function *mag()* has been added to Table 2 for the magnitude of a complex dual number. The function is useful for the purpose of numerical implementation of algorithms.

Table 2. COMPLEX DUAL FUNCTIONS WITH $\check{w} = z_1 + \varepsilon z_2 = x + iy + \varepsilon u + i\varepsilon v$, $\check{w}_1 = z_{11} + \varepsilon z_{12}$, AND $\check{w}_2 = z_{21} + \varepsilon z_{22}$

Functions	Semantics
$\mathrm{abs}(\check{w})$	$\mathrm{sqrt}(x^2+y^2) + \varepsilon \frac{xu+yv}{\mathrm{sqrt}(x^2+y^2)}$
$\mathrm{mag}(\check{w})$	$\mathrm{sqrt}(x^2+y^2)$
$\mathrm{conjugate}(\check{w})$	$\overline{z_1} + \varepsilon\, \overline{z_2}$
$\mathrm{sqrt}(\check{w})$	$\mathrm{sqrt}(z_1) + \varepsilon\, \frac{z_2}{2\mathrm{sqrt}(z_1)}$
$\exp(\check{w})$	$\exp(z_1)(1 + \varepsilon\, z_2)$
$\log(\check{w})$	$\log(z_1) + \varepsilon\, \frac{z_2}{z_1}$
$\mathrm{log10}(\check{w})$	$\frac{\log(\check{w})}{\log(10)}$
$\mathrm{pow}(\check{w}_1, \check{w}_2)$	$\check{w}_1^{\check{w}_2} = z_{11}^{z_{21}} + \varepsilon\,(z_{12}z_{21}z_{11}^{z_{21}-1} + z_{22}z_{11}^{z_{21}}\log(z_{11}))$
$\sin(\check{w})$	$\sin(z_1) + \varepsilon\, z_2\cos(z_1)$
$\cos(\check{w})$	$\cos(z_1) - \varepsilon\, z_2\sin(z_1)$
$\tan(\check{w})$	$\tan(z_1) + \varepsilon\, \frac{z_2}{\cos(z_1)\cos(z_1)}$
$\mathrm{asin}(\check{w})$	$\mathrm{asin}(z_1) + \varepsilon\, \frac{z_2}{\mathrm{sqrt}(1-z_1^2)}$
$\mathrm{acos}(\check{w})$	$\mathrm{acos}(z_1) - \varepsilon\, \frac{z_2}{\mathrm{sqrt}(1-z_1^2)}$
$\mathrm{atan}(\check{w})$	$\mathrm{atan}(z_1) + \varepsilon\, \frac{z_2}{1+z_1^2}$
$\sinh(\check{w})$	$\sinh(z_1) + \varepsilon\, z_2\cosh(z_1)$
$\cosh(\check{w})$	$\cosh(z_1) + \varepsilon\, z_2\sinh(z_1)$
$\tanh(\check{w})$	$\tanh(z_1) - \varepsilon\, \frac{z_2}{\cosh(z_1)\cosh(z_1)}$
$\mathrm{asinh}(\check{w})$	$\mathrm{asinh}(z_1) + \varepsilon\, \frac{z_2}{\mathrm{sqrt}(z_1^2+1)}$
$\mathrm{acosh}(\check{w})$	$\mathrm{acosh}(z_1) - \varepsilon\, \frac{z_2}{\mathrm{sqrt}(z_1^2-1)}$
$\mathrm{atanh}(\check{w})$	$\mathrm{atanh}(z_1) + \varepsilon\, \frac{z_2}{1-z_1^2}$

ANALYSIS OF AN RCCC SPATIAL MECHANISM USING A DUAL POLYNOMIAL

In this section, we will demonstrate how dual polynomials and complex dual numbers can be used for analysis of RCCC spatial mechanisms. A schematic of the RCCC spatial mechanism is shown in Figure 1. A second-order dual polynomial can be developed to solve $\widehat{\theta}_4$ for a given θ_1. This dual polynomial is developed from the input-displacement equation for an RCCC. This equation is developed as follows. Using dual numbers, the transformation matrix which transforms the coordinate system from

Table 3. PROPERTIES FOR COMPLEX DUAL NUMBERS OF $\check{w} = x + iy + \varepsilon u + i\varepsilon v$, $\check{w}_1 = x_1 + iy_1 + \varepsilon u_1 + i\varepsilon v_1$ AND $\check{w}_2 = x_2 + iy_2 + \varepsilon u_2 + i\varepsilon v_2$

1.	$\overline{\check{w}_1 + \check{w}_2} = \overline{\check{w}_1} + \overline{\check{w}_2}$
2.	$\overline{\check{w}_1\check{w}_2} = \overline{\check{w}_1}\,\overline{\check{w}_2}$
3.	$\overline{\check{w}_1/\check{w}_2} = \overline{\check{w}_1}/\overline{\check{w}_2}$
4.	$\check{w}\overline{\check{w}} = \check{w}^2$. If $\check{w} \neq 0$,
	$\mathrm{dual}(\check{w}) \neq 0$, and $\mathrm{imag\_dual}(\check{w}) \neq 0$, we have
	$\check{w}^{-1} = \overline{\check{w}}/\lvert\check{w}\rvert^2$
5.	$\check{w} = \overline{\check{w}}$ if and only if $y = v = 0$
6.	real $(\check{w})$ + dual $(\check{w}) = \frac{\check{w}+\overline{\check{w}}}{2}$, and
	imag $(\check{w})$+ imag_dual$(\check{w}) = \frac{\check{w}-\overline{\check{w}}}{2i}$
7.	$\overline{\overline{\check{w}}} = \check{w}$.
8.	$\lvert\check{w}_1\check{w}_2\rvert = \lvert\check{w}_1\rvert\lvert\check{w}_2\rvert$.
9.	If real $(\check{w}_1)$ or imag $(\check{w}_1) \neq 0$, then $\lvert\check{w}_1/\check{w}_2\rvert = \lvert\check{w}_1\rvert/\lvert\check{w}_2\rvert$.
10.	$\lvert\overline{\check{w}_1}\rvert = \lvert\check{w}_1\rvert$.
11.	$\sin^2\check{w} + \cos^2\check{w} = 1$
12.	$\sin(\check{w}_1 + \check{w}_2) = \sin\check{w}_1\cos\check{w}_2 + \cos\check{w}_1\sin\check{w}_2$,
	$\cos(\check{w}_1 + \check{w}_2) = \cos\check{w}_1\cos\check{w}_2 - \sin\check{w}_1\sin\check{w}_2$.
13.	If a complex dual number $\check{w} = x + iy + \varepsilon u + i\varepsilon v$
	is a root of a dual polynomial, then the conjugate
	$\overline{\check{w}} = x - iy + \varepsilon u - i\varepsilon v$ is also a root.

$X_iY_iZ_iO_i$ to $X'Y'Z'O'$ in Figure 2 is defined as

$$\widehat{\mathbf{\Theta}}_i = \begin{bmatrix} \cos\widehat{\theta}_i & -\sin\widehat{\theta}_i & 0 \\ \sin\widehat{\theta}_i & \cos\widehat{\theta}_i & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{54}$$

where dual angle $\widehat{\theta}_i = \theta_i + \varepsilon d_i$. The transformation $\widehat{\mathbf{\Lambda}}_i$ of the coordinate system from $X'Y'Z'O'$ to $X_{i+1}Y_{i+1}Z_{i+1}O_{i+1}$ in Figure 2 is defined as

$$\widehat{\mathbf{\Lambda}}_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\widehat{\alpha}_i & -\sin\widehat{\alpha}_i \\ 0 & \sin\widehat{\alpha}_i & \cos\widehat{\alpha}_i \end{bmatrix}, \tag{55}$$

where dual angle $\widehat{\alpha}_i = \alpha_i + \varepsilon a_i$. A 3×3 dual matrix can be also used to describe the transformation between the co-

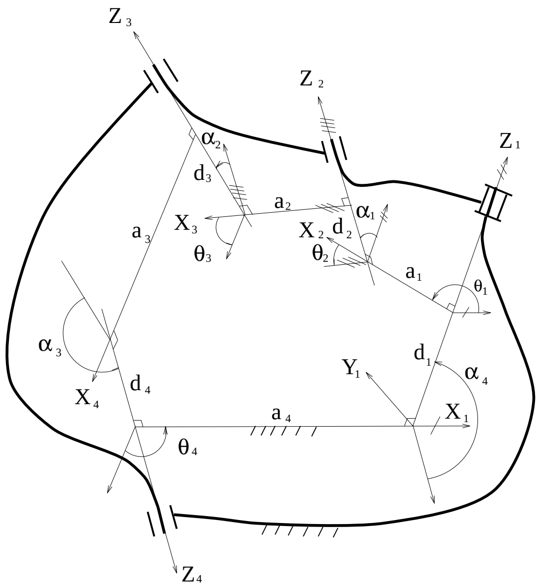

Figure 1. THE RCCC MECHANISM SCHEMATIC

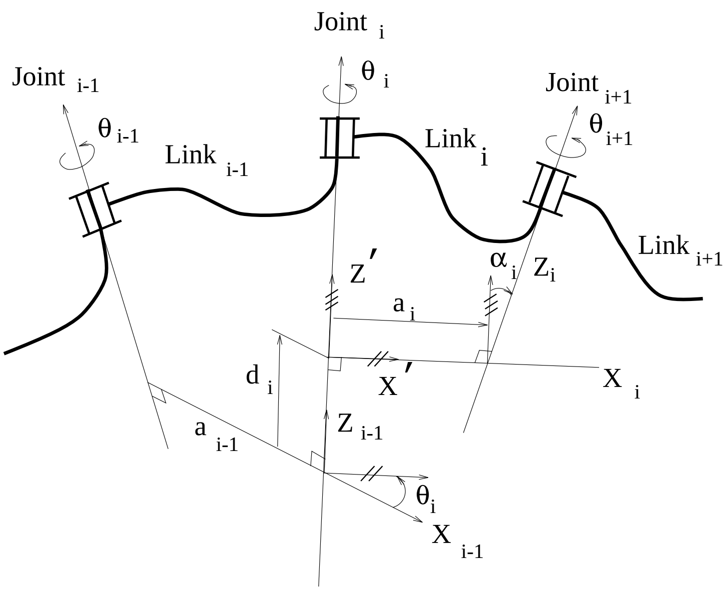

Figure 2. THE DENAVIT-HARTENBERG PARAMETERS FOR ADJACENT LINKS

ordinate systems $X_iY_iZ_iO_i$ and $X_{i+1}Y_{i+1}Z_{i+1}O_{i+1}$ shown in Figure 2. This dual transformation matrix can be expressed as (Gupta, 1989; Cheng, 1994)

$$\widehat{\mathbf{A}}_i(\widehat{\theta}_i) = [\widehat{\mathbf{\Theta}}_i][\widehat{\mathbf{\Lambda}}_i] = \begin{bmatrix} \cos\widehat{\theta}_i & -\sin\widehat{\theta}_i\cos\widehat{\alpha}_i & \sin\widehat{\theta}_i\sin\widehat{\alpha}_i \\ \sin\widehat{\theta}_i & \cos\widehat{\theta}_i\cos\widehat{\alpha}_i & -\cos\widehat{\theta}_i\sin\widehat{\alpha}_i \\ 0 & \sin\widehat{\alpha}_i & \cos\widehat{\alpha}_i \end{bmatrix}. \tag{56}$$

Matrix equation (56) is called dual Denavit-Hartenberg parameters for the adjacent links. In Figure 1, dual angles $\widehat{\alpha}_i = \alpha_i + \varepsilon a_i$ (i = 1, 2, 3, 4) and d_1 are given link parameters, and θ_i, α_i, a_i, and d_i follow the convention depicted in Figure 2. The revolute axis Z_1 and cylindrical axis Z_4 are fixed in space. Coordinate systems $O_1X_1Y_1Z_1$ and $O_4X_4Y_4Z_4$ are attached to the input and output links, respectively. The joint variable θ_1 is the input, the dual joint variable $\widehat{\theta}_4 = \theta_4 + \varepsilon d_4$ is the output. The other two dual joint variables are $\widehat{\theta}_2 + \varepsilon d_2$ and $\widehat{\theta}_3 = \theta_3 + \varepsilon d_3$. The displacement analysis of this mechanism is that, given link parameters $\widehat{\alpha}_i = \alpha_i + \varepsilon\alpha_i (i = 1, 2, 3, 4)$ and d_1, determine the output $\widehat{\theta}_4$ and joint variables $\widehat{\theta}_2$ and $\widehat{\theta}_3$ in terms of the input angle θ_1.

The solution of displacements of output and moving joints can be derived as follows. According to screw motion matrices, we can write the loop closure equation for the RCCC mechanism shown in Figure 1 as

$$\widehat{\mathbf{\Theta}}_3\widehat{\mathbf{\Lambda}}_3\widehat{\mathbf{\Theta}}_4\widehat{\mathbf{\Lambda}}_4\widehat{\mathbf{\Theta}}_1\widehat{\mathbf{\Lambda}}_1\widehat{\mathbf{\Theta}}_2\widehat{\mathbf{\Lambda}}_2 = \mathbf{I} \tag{57}$$

where transformation matrix $\widehat{\mathbf{\Theta}}_i$ is given in equation (54), transformation matrix $\widehat{\mathbf{\Lambda}}_i$ is given in equation (55) and $\mathbf{I}$ is a 3×3 identity matrix. Equation (57) can be reformulated as

$$\widehat{\mathbf{\Lambda}}_3\widehat{\mathbf{\Theta}}_4\widehat{\mathbf{\Lambda}}_4\widehat{\mathbf{\Theta}}_1\widehat{\mathbf{\Lambda}}_1 = \widehat{\mathbf{\Theta}}_3^T\widehat{\mathbf{\Theta}}_2^T\widehat{\mathbf{\Lambda}}_2^T = \mathbf{I} \tag{58}$$

where $\widehat{\mathbf{\Theta}}_i^T$ and $\widehat{\mathbf{\Lambda}}_i^T$ are transposes of $\widehat{\mathbf{\Theta}}_i$ and $\widehat{\mathbf{\Lambda}}_i$, respectively. After expanding the above matrix equation, element (3,3) gives us the following input-output relation of the mechanism

$$\widehat{A}\sin\widehat{\theta}_4 + \widehat{B}\cos\widehat{\theta}_4 + \widehat{C} = 0, \tag{59}$$

where

$$\widehat{A} = \sin\widehat{\alpha}_1\sin\widehat{\alpha}_3\sin\widehat{\theta}_1 \tag{60}$$

$$\widehat{B} = -\sin\widehat{\alpha}_3(\cos\widehat{\alpha}_1\sin\alpha_4 + \sin\widehat{\alpha}_1\cos\widehat{\alpha}_4\cos\widehat{\theta}_1) \tag{61}$$

$$\widehat{C} = \cos\widehat{\alpha}_3(\cos\widehat{\alpha}_1\cos\alpha_4 - \sin\widehat{\alpha}_1\sin\widehat{\alpha}_4\cos\widehat{\theta}_1) - \cos\widehat{\alpha}_2. \tag{62}$$

Substituting $\check{\chi} = \tan(\widehat{\theta}_4/2)$ into equation (59) gives the dual polynomial equation

$$(\widehat{C} - \widehat{B})\check{\chi}^2 + 2\widehat{A}\check{\chi} + (\widehat{C} + \widehat{B}) = 0. \tag{63}$$

The second-order dual polynomial in equation (63) can be solved by the formula for the quadratic equation

$$\check{\chi} = \frac{-\widehat{A} \pm \sqrt{\widehat{A}^2 + \widehat{B}^2 - \widehat{C}^2}}{\widehat{C} - \widehat{B}}. \tag{64}$$

Note that $\check{\chi}$ is a complex dual number so that solutions can be obtained if the real part of the dual value inside the radial is negative. Valid solutions to $\widehat{\theta}_4$ within the range of motion shall be proper dual numbers.

For an RCCC mechanism with the following specifications (Sugimoto, et al, 1982, Cheng and Thompson, 1996)

$\widehat{\alpha}_1 = 90° + \varepsilon 2.0$ cm, $\widehat{\alpha}_2 = 60° + \varepsilon 1.5$ cm, $\widehat{\alpha}_3 = 60° + \varepsilon 1.0$ cm, $\widehat{\alpha}_4 = 90° + \varepsilon 3.0$ cm, and $d_1 = 2.5$ cm.

Solutions of output displacement with respect to the input angle θ_1 are shown in Figures 3 and 4. We will examine the solutions to dual polynomial equation (63) for different values of the input angle θ_1. As shown in Figures 3 and 4, there is no solution of proper dual number for $\theta_1 = 15°$. The dual polynomial equation (63) for $\theta_1 = 15°$ becomes

$$\begin{aligned}(-0.982962889059918 - \varepsilon 1.782521661097981)\check{\chi}^2 + \\ (0.448287716171002 + \varepsilon 4.441400546646387)\check{\chi} + \\ (-0.982963039276556 + \varepsilon 6.700677605023184) = 0.\end{aligned} \tag{65}$$

The solutions to the polynomial equation (65) are

$$\begin{aligned}\check{\chi}_1 &= 0.228028775375387 + i0.973654314826659 + \\ &\quad \varepsilon 3.813620077252192 + i\varepsilon 3.538730745820137\end{aligned} \tag{66}$$

$$\begin{aligned}\check{\chi}_2 &= 0.228028775375387 - i0.973654314826659 + \\ &\quad \varepsilon 3.813620077252192 - i\varepsilon 3.538730745820136.\end{aligned} \tag{67}$$

The roots $\check{\chi}_1$ and $\check{\chi}_2$ are dual complex conjugates, which also demonstrates property (13) in Table 3 — if one of the roots to a dual polynomial equation is dual complex, then its conjugate is also a root.

At $\theta_1 = 130°$, the polynomial of equation (63) becomes

$$\begin{aligned}(-0.178606215014662 + \varepsilon 1.637883549652602)\check{\chi}^2 + \\ (1.326827944752881 - \varepsilon 2.017307097331205)\check{\chi} + \\ (-0.178606242309309 + \varepsilon 1.761963151545676) = 0.\end{aligned} \tag{68}$$

The solutions of equation (68) are

$$\begin{aligned}\check{\chi}_1 &= 0.137143265552066 + i0.0 \\ &\quad -\varepsilon 1.186463419910504 + i\varepsilon 0.0\end{aligned} \tag{69}$$

$$\begin{aligned}\check{\chi}_2 &= 7.291646066577051 + i0.0 \\ &\quad +\varepsilon 58.016427275113223 + i\varepsilon 0.0.\end{aligned} \tag{70}$$

As expected, both of these two roots are proper dual numbers. The imaginary and imaginary dual parts of $\check{\chi}_1$ and $\check{\chi}_2$ are identically zero. The corresponding displacement value $\widehat{\theta}_4$ can be obtained by $\widehat{\theta}_4 = 2\text{atan}(\check{\chi})$. They are $\widehat{\theta}_4 = \theta_4 + \varepsilon d_4 = 15.642° - \varepsilon 2.329$ and $\widehat{\theta}_4 = \theta_4 + \varepsilon d_4 = 164.382° + \varepsilon 2.142$ as shown in Figures 3 and 4.

SINGULARITY ANALYSIS OF AN RCCC SPATIAL MECHANISM USING DUAL POLYNOMIALS AND COMPLEX DUAL NUMBERS

The input-output dual displacement equation can provide useful information about the mechanism. For example, if there exists a solution of proper dual numbers for dual polynomial equation (63) when the input angle θ_1 changes in the complete motion range of 0 to 2π, the input link is a fully rotatable crank.

It is known that singularities in motion occur if an overconstrained spatial mechanism with the driving input joint locked becomes movable (Litvin, et al., 1986). The singularity of this kind is called *stationary configuration* by Hunt (1978). The idea can also be applied to singularity analysis of open-chain robot manipulators by holding the end-effector of a robot fixed (Lipkin and Duffy, 1982; Lai and Yang, 1986). The mechanism is also at a singular configuration when a displacement value goes to infinity. The input-output dual displacement equation can be conveniently used for singularity analysis of spatial mechanisms. Based on Theorems 1 and 2, the idea of identifying singularities using the input-output dual displacement equation is simple. If the roots obtained from equation (30) contain identical real roots, the mechanism is at a singular configuration. If the determinant (33) is zero, corresponding to the identical real roots, an internal degree of freedom is introduced at this singular configuration, because the dual component $\grave{x}$ of the translation motion can be any value. If the determinant (33) is not zero, the translational displacement corresponding to the dual part of the solution becomes infinity. In practice, infinity is not achievable in a mechanism. Similarly, due to round-off errors in numerical computation, a very large value rather than infinity may be obtained in the program near a singularity. The idea will

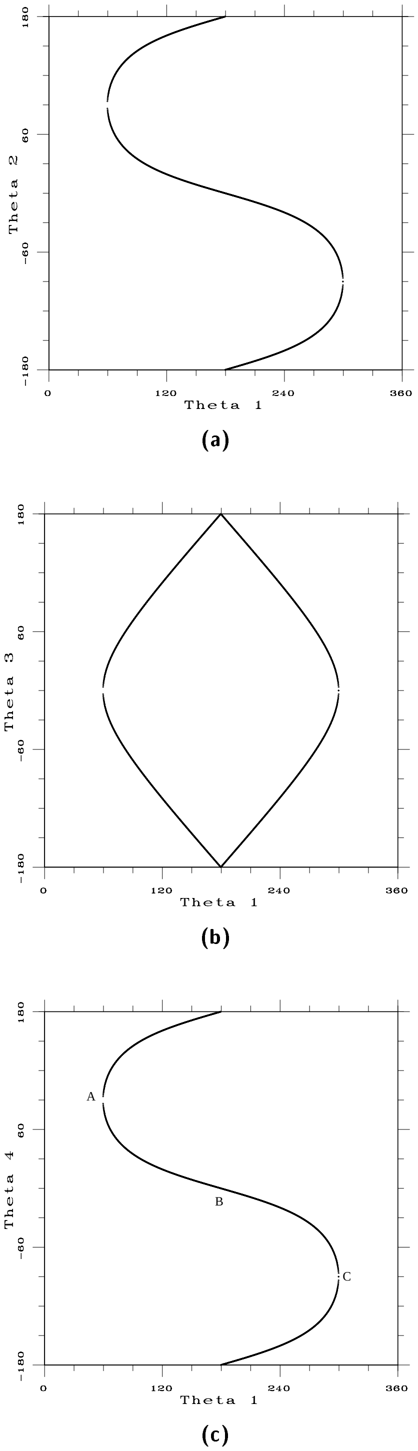

Figure 3. GRAPHICAL OUTPUT FOR JOINT VARIABLES θ_2, θ_3, and θ_4 FOR COMPLETE RANGE OF INPUT ANGLE θ_1 FOR AN RCCC SPATIAL MECHANISM

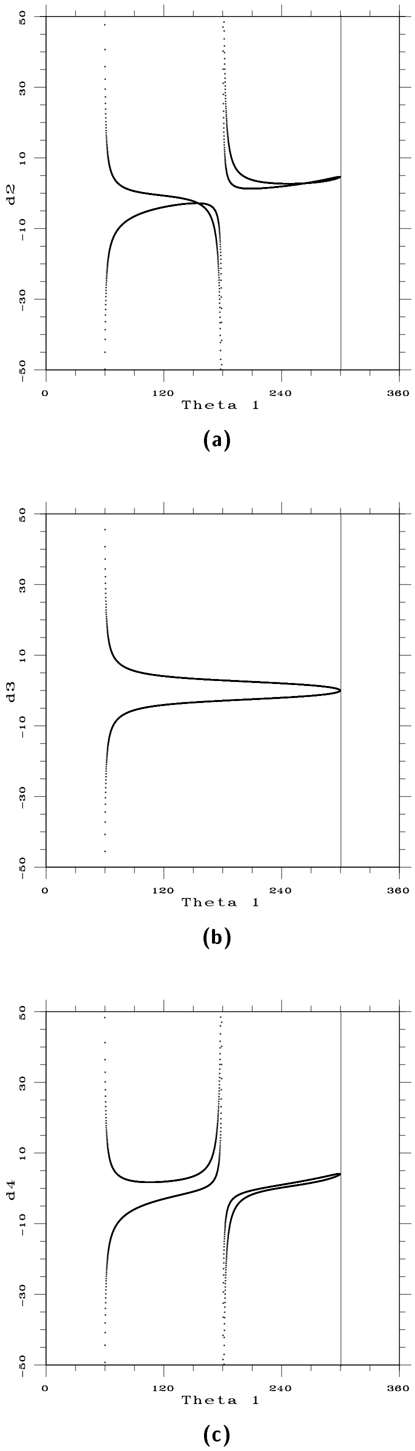

Figure 4. GRAPHICAL OUTPUT FOR JOINT VARIABLES d_2, d_3, AND d_4 FOR COMPLETE RANGE OF INPUT ANGLE θ_1 FOR AN RCCC SPATIAL MECHANISM

be illustrated through the singularity analysis of the RCCC spatial mechanism.

The RCCC spatial mechanism with specifications given in the previous section is used for singularity analysis. Graphical solutions to this RCCC spatial mechanism in Figures 3 and 4 show that singularity positions occur at points A, B, and C near $\theta_1 = 60°$, 180°, and 300°, respectively. The singularities approximately at $\theta_1 = 60°$, 180°, and 300° are known to exist because they are bifurcation positions (Sugimoto et. al, 1982). Because the singularity analysis is sensitive to numerical error, we use double precision in computer programs and retain the significant digits of numerical values in this section.

The plots in Figure 4 show that both displacement values d_2 and d_4 go to infinity whereas θ_2 and θ_4 are at bifurcation positions when θ_1 is near 60°. The second-order dual polynomial equation (63) at $\theta_1 = 60.0000050552531°$ becomes

$$\begin{aligned}(-0.75000007004261 + \varepsilon 5.84567119430922)\check{\chi}^2 + \\ (1.50000002547003 + \varepsilon 3.03108886177757)\check{\chi} + \\ (-0.74999995542742 - \varepsilon 0.21650636381310) = 0 \end{aligned} \quad (71)$$

The roots of this polynomial equation determined from equation (64) are

$$\check{\chi}_1 = 1.000000036673693 - \varepsilon 145294948.40862387 \quad (72)$$
$$\check{\chi}_2 = 1.000000116146557 + \varepsilon 145294951.87272557 \quad (73)$$

with corresponding roots of $\theta_4 = (90° - \varepsilon 145294948.4)$ and $\theta_4 = (90° + \varepsilon 145294951.8)$. And the value under the radical in equation (64) during the formation of the two roots is $(1.0 \times 1.0^{-15} + \varepsilon 6.495190032084958)$. In this case, the non-zero value of the determinant (12) is 0.43301, while the value of $2\acute{d}_2\acute{x}_1 + \acute{d}_1$ equal to $1 * 10^{-28}$ is considered as zero. Therefore, the dual part of the roots approaches $\pm\infty$ as described in Theorem 1. The singularity is caused by the infinite displacement of the two joint values d_2 and d_4.

The plots in Figure 4 show that both displacement values d_2 and d_4 go to infinity at $\theta_1 = 180°$. That is position B in Figure 3(c). The second-order dual polynomial in $\check{\chi}$ at $\theta_1 = 180°$ is

$$\begin{aligned}(-1.99840129532252 * 10^{-15} + \varepsilon 1.29903819390746)\check{\chi}^2 + \\ (1.52820250249853 * 10^{-7} - \varepsilon 4.33012685716566)\check{\chi} + \\ (-1.99840159332803 * 10^{-15} - \varepsilon 0.43301258425111) = 0 \end{aligned} \quad (74)$$

When the polynomial is solved, using equation (64), the two roots produced are

$$\check{\chi}_1 = 0.000000013245478 + \varepsilon 2943116.267416161 \quad (75)$$

$$\begin{aligned}\check{\chi}_2 = 76471252.56076711 - \\ \varepsilon 4970927206768313158 8608.0\end{aligned} \quad (76)$$

with corresponding joint values of $\widehat{\theta}_4 = (0° + \varepsilon 5886232)$ and $\widehat{\theta}_4 = (180° - \varepsilon 17000863)$. And the value inside the radical in equation (64) is $(5.8385071913 * 10^{-15} + \varepsilon 3.30865532 * 10^{-7})$. The singularity at $\theta_1 = 180°$ is caused by the dual part of each solution of $\check{\chi}$ going to infinity . Gupta (1989) pointed out that there is a singularity if the value inside the radical in equation (64) is zero. This is a sufficient, but not necessary, condition for the existence of the singularity, as demonstrated at the singular position $\theta_1 = 60°$.

By looking at the plots in Figure 3, it can be seen that two branches of θ_4 separate approximately at $\theta_1 = 300°$, which is the bifurcation point C in Figure 3(c). The second-order dual polynomial in $\check{\chi}$ at $\theta_1 = 300.0000050552535°$ is

$$\begin{aligned}(-0.750000070042609 - \varepsilon 3.680607335601163)\check{\chi}^2 + \\ (-1.500000025470035 + \varepsilon 1.299037642464895)\check{\chi} + \\ (-0.749999955427426 - \varepsilon 2.381569891655609) = 0.\end{aligned} \quad (77)$$

When the polynomial is solved, using equation (64), the two roots produced are

$$\check{\chi}_1 = -1.000000012029747 - \varepsilon 2.810854535933114 \quad (78)$$
$$\check{\chi}_2 = -1.000000140790504 + \varepsilon 10.893758735938531 \quad (79)$$

with corresponding solutions of $\widehat{\theta}_4 = (-90° - \varepsilon 2.81)$ and $\widehat{\theta}_4 = (-90° - \varepsilon 10.89)$. When the coefficients of polynomial equation (77) are each truncated off to 15 decimal places, the solution becomes

$$\check{\chi}_1 = -1.000000010514707 - \varepsilon 2.653309173068425 \quad (80)$$
$$\check{\chi}_2 = -1.000000142305543 + \varepsilon 10.736213373073841 \quad (81)$$

The value inside the radical in equation (64) is $(2.3314683517 * 10^{-15} - \varepsilon 4.962982828180 * 10^{-7})$. If the numerical procedure described in section is used to solve equation (77), the two roots produced are

$$\check{\chi}_1 = -1.000000010305176 - \varepsilon 2.632088955020015 \quad (82)$$
$$\check{\chi}_2 = -1.000000142515075 + \varepsilon 10.714993141739533. \quad (83)$$

Note that the real parts of $\check{\chi}_1$ and $\check{\chi}_2$ are nearly identical and the determinant (12) of value $1.05201 * 10^{-6}$ nearly equals zero. Therefore, there exists infinite solutions for the dual part of $\check{\chi}$ in equation (77). Table 4 shows how $\widehat{\theta}_2$ and $\widehat{\theta}_3$ behave if θ_1 is fixed at $\theta_1 = 300.0000050552535°$, θ_4 is fixed at $\theta_4 = -90.000003345030976°$, and d_4 varies from −7 cm to 7 cm. The table shows that θ_2 and θ_3 stay constant whereas d_2 and d_3 move proportionally to the motion of d_4.

Table 4. JOINT VALUES $\widehat{\theta}_2$ AND $\widehat{\theta}_3$ AT THE SINGULAR POSITION $\theta_1 = 300°$ AND $\widehat{\theta}_4 = -90°$ WHEN d_4 VARIES FROM -7 CM TO 7 CM FOR AN RCCC MECHANISM.

d_4	θ_2	d_2	θ_3	d_3	$\widehat{\zeta}$
-7.0	-90.000011	-6.422	0.000004	11.041	$0.0{-}\varepsilon 1.72 \times 10^{-6}$
-6.0	-90.000011	-5.422	0.000004	10.041	$0.0{-}\varepsilon 1.63 \times 10^{-6}$
-5.0	-90.000011	-4.422	0.000004	9.041	$0.0{-}\varepsilon 1.53 \times 10^{-6}$
-4.0	-90.000011	-3.422	0.000004	8.041	$0.0{-}\varepsilon 1.43 \times 10^{-6}$
-3.0	-90.000011	-2.422	0.000004	7.041	$0.0{-}\varepsilon 1.34 \times 10^{-6}$
-2.0	-90.000011	-1.422	0.000004	6.041	$0.0{-}\varepsilon 1.24 \times 10^{-6}$
-1.0	-90.000011	-0.422	0.000004	5.041	$0.0{-}\varepsilon 1.14 \times 10^{-6}$
0.0	-90.000011	0.577	0.000004	4.041	$0.0{-}\varepsilon 1.05 \times 10^{-6}$
1.0	-90.000011	1.577	0.000004	3.041	$0.0{-}\varepsilon 9.55 \times 10^{-7}$
2.0	-90.000011	2.577	0.000004	2.041	$0.0{-}\varepsilon 8.58 \times 10^{-7}$
3.0	-90.000011	3.577	0.000004	1.041	$0.0{-}\varepsilon 7.62 \times 10^{-7}$
4.0	-90.000011	4.577	0.000004	0.041	$0.0{-}\varepsilon 6.65 \times 10^{-7}$
5.0	-90.000011	5.577	0.000004	-0.958	$0.0{-}\varepsilon 5.69 \times 10^{-7}$
6.0	-90.000011	6.577	0.000004	-1.958	$0.0{-}\varepsilon 4.72 \times 10^{-7}$
7.0	-90.000011	7.577	0.000004	-2.958	$0.0{-}\varepsilon 3.76 \times 10^{-7}$

The residual errors $\widehat{\zeta}$ in Table 4 are obtained when $\check{\chi}_1$ and $\check{\chi}_2$ are substituted into equation (63). At this singular configuration, when the driver input link 1 is locked at $\theta_1 = 300°$, the system still has mobility. The system can have translational motion along d_2, d_3, and d_4 with fixed joint angles for θ_2, θ_3, and θ_4. The graphs that show d_2, d_3, and d_4 as θ_1 is incremented in Figure 4 reflect the additional local degree of freedom. This result matches the result reported in (Sugimoto, et al., 1982).

CONCLUSIONS

Dual polynomials and their analytical and numerical solutions in both domains of dual numbers and complex dual numbers as well as their applications in analysis of spatial mechanisms have been presented in this paper. For a dual polynomial equation (28), the solution depends on its real component (30) and determinant (33). If there exist multiple identical roots to equation (30), the dual polynomial is singular. For a singular dual polynomial equation, there are infinite solutions if the determinant (33) is zero; the dual part of solution is infinity if the determinant (33) is not zero.

For analytical and numerical treatment of dual polynomial equation (28), complex dual numbers which form a commutative ring are for the first time introduced in this paper. Complex dual operations and functions are defined within the domain of complex dual numbers. Like solving real polynomial equations in the complex domain, the dual polynomial equation (28) can be solved within the domain of complex dual numbers. Many properties of complex numbers can be applied to complex dual numbers. For example, if a complex dual number is a root of a dual polynomial equation, then its conjugate is also a root. Two numerical schemes, based on Laguerre's formulation and solutions of eigenvalues of a dual polynomial companion matrix in CHLAPACK — a modified mathematical software package LAPACK with over 10,000 line code in C^H, for solving dual polynomials in the domain of complex dual numbers are designed and implemented in the C^H language environment. The accuracy and efficiency of these two numerical algorithms implemented in C^H are comparable. The built-in dual data type in C^H significantly simplifies the implementation of complex dual operations and functions for analysis of spatial mechanisms.

Theorems and computer programs have been applied to analysis of RCCC spatial mechanism through its second-order input-output dual polynomial displacement equations. Of particular interest is that the singularities of an input-output dual displacement equation of a mechanism correspond to its singularity configurations, which provides a clear physical inside into the nature of singularity. The C^H language environment as well as all numerical and symbolic computer programs for solving dual polynomial equations based on two different formulation schemes and for analysis of the RCCC mechanism described in this paper are available free on the WWW at the URL address `http://iel.ucdavis.edu`. It is hoped that complex dual numbers and these computer programs will find their applications in a wide range of engineering and science.

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