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Derivation of the Explicit Solution of the Inverse Involute Function and its Applications

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Abstract

The involute function $\epsilon = \tan\phi - \phi$ or $\epsilon = \text{inv}\phi$, and the inverse involute function $\phi = \text{inv}^{-1}(\epsilon)$ arise in the tooth geometry calculations of involute gears, involute splines, and involute serrations. In this paper, the explicit series solutions of the inverse involute function are derived by perturbation techniques in the ranges of $|\epsilon| < 1.8$, $1.8 < |\epsilon| < 5$, and $|\epsilon| > 5$. These explicit solutions are compared with the exact solutions, and the expressions for estimated errors are also developed. Of particular interest in the applications are the simple expansion $\phi = \text{inv}^{-1}(\epsilon) = (3\epsilon)^{1/3} - 2\epsilon/5$ which gives the angle $\phi(< 45°)$ with error less than 1.0% in the range of $\epsilon < 0.215$, and the economized asymptotic series expansion $\phi = \text{inv}^{-1}(\epsilon) = 1.440859\epsilon^{1/3} - 0.3660584\epsilon$ which gives ϕ with error less than 0.17% in the range of $\epsilon < 0.215$. The four, seven, and nine term series solutions of $\phi = \text{inv}^{-1}(\epsilon)$ are shown to have error less than 0.0018%, $4.89 * 10^{-6}$%, and $2.01 * 10^{-7}$% in the range of $\epsilon < 0.215$, respectively. The computation of the series solution of the inverse involute function can be easily performed by using a pocket calculator, which should lead to its practical applications in the design and analysis of involute gears, splines, and serrations.

1. Introduction

The involute curve is most widely used for the tooth shape of gears, splines, and serrations. In Fig.1, the involute curve BC is generated with respect to the base circle with radius r_b such that the length of the normal AP equals the length of the arc AB. We can derive the parametric expression for the involute curve BC as follows (Mabie and Reinholtz, 1987; Paul, 1979; Shigley and Uicker, 1980).

$$\begin{cases} r = \dfrac{r_b}{\cos\phi} \\ \epsilon = \text{inv}\phi = \tan\phi - \phi \end{cases} \tag{1}$$

If the pressure angle ϕ is known, invϕ can be readily determined. But, in many applications, the values of $\epsilon = \text{inv}\phi$ is usually known while the pressure angle ϕ is to be found. This problem arises quite often in the gear tooth geometry calculations such as in the determinations of the non-standard center distance, non-standard pin dimensions, and outside radius of the pinion at which the tooth becomes pointed, offsets for cutting non-standard gears with teeth of equal strength, and minimum achievable quality in generation processes and maximum profile heights of spur gears (Green and Mabie, 1980; Mabie and Reinholtz, 1987; Sankaranarayanasamy and Shunmugam, 1988; Shigley and Uicker, 1980); as well as in the measurement of threads over wires. The calculations of the inverse involute function are also involved in the sevolute function for involute splines and serrations such as in finding the measurement between pins for the maximum actual space width of an internal spline and for the minimum actual space width of an external spline (Amiss et al., 1984, pp. 147-154; Ryffel, 1984, pp. 895-927). The sevolute function is defined as the difference between the secant of an angle and the involute of the angle as follows (Ryffel, 1984, p. 86)

$$\text{sev}\phi = \sec\phi - \text{inv}\phi \tag{2}$$

Obviously, no exact solution for $\phi = \text{inv}^{-1}(\epsilon)$ can be expressed explicitly in terms of elementary functions of ϵ. Due to the lack of existence of the explicit solution of the inverse involute function, some mechanism textbooks (Shigley, 1969, for $0° < \phi < 45°$; Shigley and Uicker, 1980; for $0° < \phi < 45°$; Mabie and Reinholtz, 1987, for $0° < \phi < 60°$) and design handbooks (Huckert, 1956, for $0° < \phi < 62°$; Parmiley, 1985, for $10° < \phi < 40°$; Ryffel, 1984, for $0° < \phi < 90°$) usually give an extensive table of $(\phi, \text{inv}\phi)$ from which $\text{inv}^{-1}(\epsilon)$ can

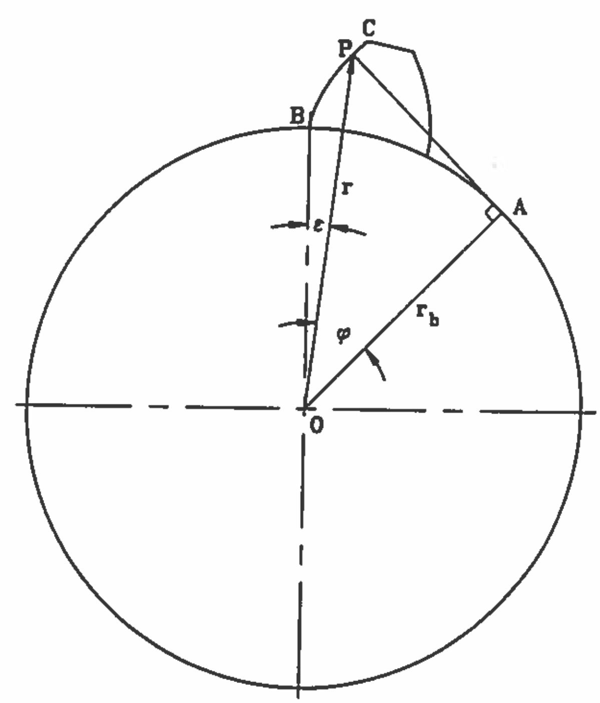

Figure 1 Base circle and involute curve.

be linearly interpolated. In order to get more accurate results, higher order interpolations are required. Another commonly used method to obtain ϕ for the given $\epsilon = \mathrm{inv}\phi$ is to use iteration methods such as the following Newton's algorithm (Gerald and Wheatley, 1984)

$$\phi_{n+1} = \phi_n + \frac{\epsilon - \tan\phi_n + \phi_n}{\tan^2\phi_n}, \quad n = 0, 1, 2, \cdots \tag{3}$$

where the value of ϵ is known, ϕ_n is the previously iterated value, and ϕ_{n+1} is the currently iterated value. It has been pointed out by Thoen (1988) that the above iterative algorithm is susceptible to divergence depending on the initial guess ϕ_0 and the range of final value of ϕ.

One way to obtain the good initial value ϕ_0 for the iteration algorithm may be through the look-up tables of $(\phi, \mathrm{inv}\phi)$. But, this is not convenient, especially, when a large number of inverse involute function calculations are required. For example, the determination of pinion-cutter offsets required to produce nonstandard spur gears with teeth of equal strength (Green and Mabie, 1980; Mabie and Reinholtz, 1987) and the determination of the minimum achievable quality in generation processes and maximum height of the profile for a given gear (Sankaranarayanasamy and Shunmugam, 1988) require a great deal of computations of the inverse involute function. The frequent manual interruptions of the computer program for the table loop-ups are too involved; and an efficient method, without susceptible to divergence, for the solution of the inverse involute function is strongly desired in such cases so that computational power of a computer can be fully utilized. An efficient and reliable method for the solution of the inverse involute function may also be desired for generating non-standard spur gears and involute splines by using CAD/CAM systems.

In this paper, the complete explicit expressions for ϕ in terms of $\epsilon = \mathrm{inv}\phi$ are derived by perturbation techniques. The value of the pressure angle ϕ is usually less than 45° in engineering applications, for which the developed explicit formula for the pressure angle ϕ has error less than $1.58 * 10^{-9}$ radian. The explicit formula of ϕ with its value less than 45° has been reported in (Cheng, 1987). Sometimes, the pressure angle ϕ may lie between $45° < \phi < 60°$, for which the error of the explicit formula for ϕ is less than $2.52 * 10^{-6}$ radian. Taking the potential accuracy of manufacturing into account, the accuracy provided by the explicit formula developed in this paper should be satisfactory for practical applications. It should be pointed out that what distinguishes the solution method presented in this paper from all currently existing methods are its noniterative and explicit natures. Since the solution of ϕ is formulated *explicitly*, unlike iteration methods, the computation of the explicit formula doesn't have divergence problem and is more convenient and efficient. Furthermore, the solution from the first two terms of the explicit formula of the inverse involute function for involute gearing applications is as accurate as that through linear interpolation of extensive tables of involute function, and the two-term formula is simple and can be remembered easily. Hence, the teaching and learning procedure of involute gearing of a mechanical design class may be facilitated by using the explicit direct and inverse involute formulas derived in this paper, and these simple formulas may make the inclusion of lengthy extensive tables of the involute function as appendices in many outstanding mechanism textbooks (Shigley, 1969, for $0° < \phi < 45°$; Shigley and Uicker, 1980; for $0° < \phi < 45°$; Mabie and Reinholtz, 1987, for $0° < \phi < 60°$) unneccessary.

2. Approximation of $\mathrm{inv}^{-1}(\epsilon)$ when $|\epsilon| \ll 1$

2.1 Approximation by Asymptotic Series

The involute pressure angle ϕ is usually less than $\pi/4$ when the involute curve is used as the tooth curve of spur gears, splines, and serrations. Since $\mathrm{inv}(\pi/4) = \tan(\pi/4) - \pi/4 \sim 0.215$, the solution obtained in this section is useful for tooth geometry calculations. For convenience, let $\epsilon = \mathrm{inv}\phi$ and $x = \phi$, then Eq.(1) becomes

$$\tan x - x = \epsilon, \qquad \epsilon \ll 1. \tag{4}$$

Since $\tan x - x = f(x)$ is an odd function of x, we can consider only the case when $\epsilon > 0$, but the result will be valid for $\epsilon < 0$ as well.

Let $y_1 = \tan x, y_2 = x + \epsilon$; then, the solution of Eq.(4) for the given ϵ is the intersection of the curves of y_1 and y_2 as shown in Fig.2 . There are infinite solutions of x corresponding to a given value of ϵ in Eq.(4). We only consider the solutions which lie within $x \in (-\pi/2, \pi/2)$. However, the following method can be easily extended for the interval $x \in (-\infty, \infty)$. When $|x| < \pi/2$, the Taylor series expansion of $\tan x$ is

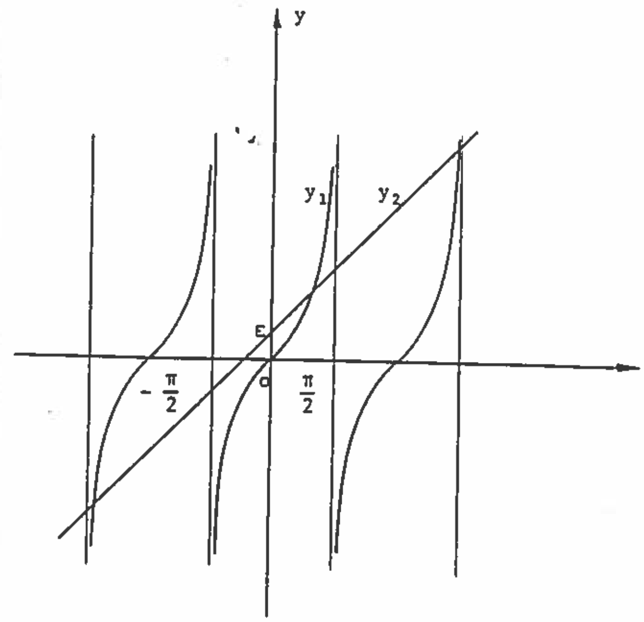

Figure 2 $y_1 = \tan x$ and $y_2 = x + \epsilon$.

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \cdots + \frac{2^{2n}(2^{2n}-1)B_n}{(2n)!}x^{2n-1} + \cdots \tag{5}$$

where B_n is the Bernoulli's constant. Substituting the expression of $\tan x$ in Eq.(5) into Eq.(4), we get

$$\frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \cdots = \epsilon. \tag{6}$$

Let the solution to the above nonlinear equation be the following asymptotic series

$$x(\epsilon) \sim \sum_{n=1}^{\infty} \delta_n(\epsilon)x_n, \qquad \text{as} \quad \epsilon \to 0^+, \tag{7}$$

where $\delta_{n+1}(\epsilon) \ll \delta_n(\epsilon)$, as $\epsilon \to 0^+$. For more information about perturbation techniques and properties of the asymptotic series, Bender and Orszag (1978), Kevorkian and Cole (1981), and Nayfeh (1981) should be consulted. In the following presentation, the first four terms of the asymptotic series of $x(\epsilon)$ in Eq.(7) are derived. Eq.(7) can be written explicitly as

$$x(\epsilon) \sim \delta_1x_1 + \delta_2x_2 + \delta_3x_3 + \delta_4x_4 + 0(\delta_5), \tag{8}$$

where δ_i's are functions of ϵ, and both δ_i and x_i are to be determined. Substituting Eq.(8) into Eq.(6), we get

$$\frac{1}{3}[\delta_1x_1 + \delta_2x_2 + \delta_3x_3 + \delta_4x_4 + 0(\delta_5)]^3 + \frac{2}{15}[\delta_1x_1 + \cdots]^5 + \frac{17}{315}[\delta_1x_1 + \cdots]^7 + \frac{62}{2835}[\delta_1x_1 + \cdots]^9 + \cdots \sim \epsilon, \tag{9}$$

or

$$\frac{1}{3}\{[(\delta_1x_1 + \delta_2x_2)^3 + 3(\delta_1x_1 + \delta_2x_2)^2(\delta_3x_3) + 0(\delta_1\delta_3^2)] + [3(\delta_1x_1 + \delta_2x_2 + \delta_3x_3)^2(\delta_4x_4)] + [0(\delta_1\delta_4^2)]\}$$
$$+ \frac{2}{15}\{[(\delta_1x_1 + \delta_2x_2)^5 + 5(\delta_1x_1 + \delta_2x_2)^4(\delta_3x_3) + 0(\delta_1^3\delta_3^2)]$$

$$
\begin{aligned}
&+5(\delta_1 x_1+\delta_2 x_2+\delta_3 x_3)^4(\delta_4 x_4)+0(\delta_1^3\delta_4^2)\} \qquad (10)\\
&+\frac{17}{315}\{[(\delta_1 x_1+\delta_2 x_2)^7+7(\delta_1 x_1+\delta_2 x_2)^6(\delta_3 x_3)+0(\delta_1^5\delta_3^2)]\\
&\qquad +0(\delta_1^6\delta_4)\}\\
&+\frac{62}{2835}\{(\delta_1 x_1)^9+0(\delta_1^8\delta_2)\}+0(\delta_1^{11})\sim\varepsilon.
\end{aligned}
$$

For brevity, we define the neglected term as $0(h)$. Expanding Eq.(10) again, we get

$$
\begin{aligned}
&\frac{1}{3}\{[(\delta_1 x_1)^3+3(\delta_1 x_1)^2(\delta_2 x_2)+3(\delta_1 x_1)(\delta_2 x_2)^2+(\delta_2 x_2)^3]\\
&\qquad +3[(\delta_1 x_1)^2+2(\delta_1 x_1)2(\delta_2 x_2)+(\delta_2 x_2)^2](\delta_3 x_3)\\
&\qquad +3(\delta_1 x_1)^2(\delta_4 x_4)+0(h)\}\\
&+\frac{2}{15}\{[(\delta_1 x_1)^5+5(\delta_1 x_1)^4(\delta_2 x_2)+10(\delta_1 x_1)^3(\delta_2 x_2)^2+0(h)]\\
&\qquad +5(\delta_1 x_1)^4(\delta_3 x_3)+0(h)\} \qquad (11)\\
&+\frac{17}{315}\{[(\delta_1 x_1)^7+7(\delta_1 x_1)^6(\delta_2 x_2)+0(h)]\\
&\qquad +7(\delta_1 x_1)^6(\delta_4 x_4)+0(h)\}\\
&+\frac{62}{2835}\{(\delta_1 x_1)^9+0(h)\}+0(h)\sim\varepsilon.
\end{aligned}
$$

Comparing the order on the both sides of Eq.(11), we can obtain the following results by using the dominant balance method (Bender and Orszag, 1978):

(i) ord(δ_1^3)=ord(ε):

$$\text{then}\quad \delta_1^3=\varepsilon, \qquad \text{or}\quad \delta_1=\varepsilon^{1/3}.$$

Setting the coefficient of δ_1^3 equal to the coefficient of ε in Eq.(11), we get

$$\frac{1}{3}x_1^3=1$$

$$\text{or}\qquad x_1=3^{2k\pi i/3}, \qquad k=0,1,2$$

According to Fig.2, we only consider the real root, hence

$$x_1=3^{1/3}$$

(ii) $\mathrm{ord}(\delta_1^2\delta_2)=\mathrm{ord}(\delta_1^5)=\mathrm{ord}(\varepsilon^{5/3})$:

$$\text{then}\quad \delta_1^2\delta_2=\delta_1^5, \quad \text{or}\quad \delta_2=\delta_1^3=\varepsilon.$$

Because the coefficient of $\varepsilon^{5/3}$ in Eq.(11) is zero, we get

$$\frac{1}{3}(3x_1^2x_2)+\frac{2}{15}x_1^5=0,$$

$$\text{or}\qquad x_2=-\frac{2}{15}x_1^3=-\frac{2}{5}.$$

(iii) $\mathrm{ord}(\delta_1\delta_2^2)=\mathrm{ord}(\delta_1^2\delta_3)=\mathrm{ord}(\delta_1^4\delta_2)=\mathrm{ord}(\delta_1^7)=\mathrm{ord}(\varepsilon^{7/3})$:

$$\text{then}\quad \delta_1^2\delta_3=\delta_1^7, \quad \text{or}\quad \delta_3=\delta_1^5=\varepsilon^{5/3}.$$

Setting the coefficient of $\varepsilon^{7/3}$ in Eq.(11) to zero, we get

$$\frac{1}{3}(3x_1x_2^2+3x_1^2x_3)+\frac{2}{15}(5x_1^4x_2)+\frac{17}{315}x_1^7=0,$$

or

$$x_3=-(x_1^{-1}x_2^2+\frac{2}{3}x_1^2x_2+\frac{17}{315}x_1^5)=\frac{9}{175}3^{2/3}$$

(iv) $\mathrm{ord}(\delta_1^3) = \mathrm{ord}(\delta_1\delta_2\delta_3) = \mathrm{ord}(\delta_1^3\delta_2^2) = \mathrm{ord}(\delta_1^4\delta_3) = \mathrm{ord}(\delta_1^6\delta_2)=\mathrm{ord}(\delta_1^2\delta_4)=\mathrm{ord}(\delta_1^9)=\mathrm{ord}(\varepsilon^3)$:

$$\text{then}\quad \delta_1^2\delta_4=\delta_1^9, \quad \text{or}\quad \delta_4=\delta_1^7=\varepsilon^{7/3}.$$

Following the same procedure as shown above, we can get

$$\frac{1}{3}(x_2^3+6x_1x_2x_3+3x_1^2x_4)+\frac{2}{15}(10x_1^3x_2^2+5x_1^4x_3)$$

$$+\frac{17}{315}(7x_1^6x_2)+\frac{62}{2835}x_1^9=0.$$

Substituting $x_1=3^{1/3}, x_2=-2/5, x_3=9*3^{2/3}/175$ into the above equation, we obtain

$$x_4=-\frac{2}{175}3^{1/3},$$

The asymptotic series $x(\varepsilon)=\mathrm{inv}^{-1}(\varepsilon)$ in Eq.(8) becomes

$$x(\varepsilon)\sim 3^{1/3}\varepsilon^{1/3}-\frac{2}{5}\varepsilon+\frac{9}{175}3^{2/3}\varepsilon^{5/3}-\frac{2}{175}3^{1/3}\varepsilon^{7/3}+\cdots \qquad (12)$$

$$\text{or}\qquad x(\varepsilon)\sim\sum_{n=1}^{\infty}a_n\varepsilon^{(2n-1)/3}, \qquad (13)$$

where $a_1=3^{1/3}, a_2=-\frac{2}{5}, a_3=\frac{9}{175}3^{2/3}, a_4=-\frac{2}{175}3^{1/3}$, etc. The coefficients of higher order terms can be derived in the same manner or by using computer symbolic manipulation software. The asymptotic series (13) has the following property

$$x(\varepsilon)-\sum_{n=1}^{N}a_n\varepsilon^{(2n-1)/3} \ll a_N\varepsilon^{(2N-1)/3}, \quad \text{as}\quad \varepsilon\to 0^+,\forall N \qquad (14)$$

or, alternatively

$$x(\varepsilon)-\sum_{n=1}^{N}a_n\varepsilon^{(2n-1)/3} \sim a_{N+1}\varepsilon^{(2N+1)/3}, \quad \text{as}\quad \varepsilon\to 0^+,\forall N. \qquad (15)$$

Therefore, we can define the asymptotic solution $x_N(\varepsilon)$ and the estimated error $E_N(\varepsilon)$ either by $a_N\varepsilon^{(2N-1)/3}$ (loose) or $a_{N+1}\varepsilon^{(2N+1)/3}$ (precise) as follows.

$$
\begin{cases}
x_2(\varepsilon)\sim 3^{1/3}\varepsilon^{1/3}-\frac{2}{5}\varepsilon\\
E_2(\varepsilon)\sim \frac{9}{175}3^{2/3}\varepsilon^{5/3}
\end{cases} \qquad (16)
$$

$$
\begin{cases}
x_9(\varepsilon)\sim 3^{1/3}\varepsilon^{1/3}-\frac{2}{5}\varepsilon+\frac{9}{175}3^{2/3}\varepsilon^{5/3}-\frac{2}{175}3^{1/3}\varepsilon^{7/3}\\
\qquad -\frac{144}{67375}\varepsilon^3+\frac{3258}{3128125}3^{2/3}\varepsilon^{11/3}-\frac{49711}{153278125}3^{1/3}\varepsilon^{13/3}\\
\qquad -\frac{1130112}{9306171875}\varepsilon^5+\frac{5169659643}{95304506171875}3^{2/3}\varepsilon^{17/3}\\
E_9(\varepsilon)\sim \frac{5169659643}{95304506171875}3^{2/3}\varepsilon^{17/3}
\end{cases} \qquad (17)
$$

In Table 1, we compare the asymptotic solutions $x_2(\varepsilon), x_9(\varepsilon)$ and the estimated errors $E_2(\varepsilon), E_9(\varepsilon)$ with their exact values. The exact solutions are obtained by evaluating $\tan x - x = \varepsilon$ directly by using an IBM 3081K with the Fortran program written in the double precision mode. Fo clarity, only the first three significant digits of the exact value ε are given in Table 1 and the following tables. From Table 1, we can see that $E_2(\varepsilon)$ is a good approximation of the exact error; i.e., $a_{N+1}\varepsilon^{(2N+1)/3}$ is a better error expression than $a_N\varepsilon^{(2N-1)/3}$ for $x(\varepsilon)-\sum_{n=1}^{N}a_n\varepsilon^{(2n-1)/3}$. In Table 1, the exact error is defined as

$$E_{\text{exact error}} = |\text{exact inv}^{-1}(\varepsilon)-\text{asymptotic inv}^{-1}(\varepsilon)| \qquad (18)$$

It should be pointed out that the exact error should be always less than the corresponding estimated error. The errors using the formula (17) is extremely small when $\phi < 10°$; the dominant errors in the fourth column of Table 1 is the round-off errors of variables because the floating-point numbers expressed in the double precision mode in the computer can only carry 15 significant digits. From Table 1, we can see that the asymptotic solution $x_2(\varepsilon)$ is a good approximation when $|\varepsilon| < 0.05$, corresponding to $\phi = 30°$; and $x_9(\varepsilon)$ matches the exact solution well. For the large value of ε, $x_9(\varepsilon)$ is still a good approximation of x even when $1 < \varepsilon \ll \infty$. By using expressions for estimated errors $E_2(\varepsilon)$ and $E_9(\varepsilon)$, one can predict the errors a prior, i.e., before calculating the asymptotic solutions.

Note that the errors of asymptotic solutions $x_2(\varepsilon)$ and $x_9(\varepsilon)$ are less than 1.0% and $2.01 * 10^{-7}$%, respectively, when $\phi < \pi/4$, and the maximum error occurs at $\phi = \pi/4$; these percent errors are the

Table 1: Comparison of $x(\epsilon) = \mathrm{inv}^{-1}(\epsilon)$ and estimated errors $E(\epsilon)$ with their exact values for $|\epsilon| \ll 1$

$\epsilon = \mathrm{inv}(x_{exact})$	x_{exact}	$x_9(\epsilon)$	E_9 (radian)		$x_2(\epsilon)$	E_2 (radian)	
			E_{exact}	$E_{estimate}$		E_{exact}	$E_{estimate}$
$1.77 * 10^{-6}$	1°	1°0.00″	$1.15 * 10^{-16}$	$2.89 * 10^{-37}$	1°0.00″	$2.78 * 10^{-11}$	$2.78 * 10^{-11}$
$2.22 * 10^{-4}$	5°	5°0.00″	$8.74 * 10^{-16}$	$2.24 * 10^{-25}$	4°59′59.98″	$8.72 * 10^{-8}$	$8.72 * 10^{-8}$
$1.79 * 10^{-3}$	10°	10°0.00″	$9.71 * 10^{-17}$	$3.10 * 10^{-20}$	9°59′59.42″	$2.83 * 10^{-6}$	$2.83 * 10^{-6}$
$6.14 * 10^{-3}$	15°	15°0.00″	$8.33 * 10^{-17}$	$3.33 * 10^{-17}$	14°59′55.47″	$2.20 * 10^{-5}$	$2.21 * 10^{-5}$
$1.47 * 10^{-2}$	20°	20°0.00″	0.00	$5.03 * 10^{-15}$	19°59′42.27″	$9.57 * 10^{-5}$	$9.66 * 10^{-5}$
$3.00 * 10^{-2}$	25°	25°0.00″	$5.75 * 10^{-15}$	$2.63 * 10^{-13}$	24°58′57.14″	$3.05 * 10^{-4}$	$3.09 * 10^{-5}$
$5.36 * 10^{-2}$	30°	30°0.00″	$2.26 * 10^{-13}$	$7.21 * 10^{-12}$	29°57′14.84″	$8.01 * 10^{-4}$	$8.19 * 10^{-4}$
$8.93 * 10^{-2}$	35°	35°0.00″	$5.97 * 10^{-12}$	$1.28 * 10^{-10}$	34°53′38.41″	$1.85 * 10^{-3}$	$1.91 * 10^{-3}$
$1.41 * 10^{-1}$	40°	40°0.00″	$1.09 * 10^{-10}$	$1.71 * 10^{-9}$	39°46′33.56″	$3.91 * 10^{-3}$	$4.08 * 10^{-3}$
$2.15 * 10^{-1}$	45°	45°0.00″	$1.58 * 10^{-9}$	$1.84 * 10^{-8}$	44°33′19.32″	$7.76 * 10^{-3}$	$8.23 * 10^{-3}$
$3.91 * 10^{-1}$	50°	50°0.00″	$1.97 * 10^{-8}$	$1.74 * 10^{-7}$			
$4.68 * 10^{-1}$	55°	55°0.05″	$2.26 * 10^{-7}$	$1.53 * 10^{-6}$			
$6.85 * 10^{-1}$	60°	60°0.52″	$2.52 * 10^{-6}$	$1.32 * 10^{-5}$			
1.01	65°	65°6.05″	$2.93 * 10^{-5}$	$1.19 * 10^{-4}$			
1.53	70°	70°1′20.70″	$3.91 * 10^{-4}$	$1.24 * 10^{-3}$			
1.82	72°	71°4′3.58″	$1.18 * 10^{-3}$	$3.37 * 10^{-3}$			

maximum error in ϕ expressed as a percentage of the range of ϕ. This accuracy is satisfactory for tooth geometry calculations.

2.2 Economization of Asymptotic Series by Chebyshev Polynomials

To improve the accuracy of the two term approximation $x_2(\epsilon)$, we can apply the process of series economization to $x(\epsilon)$ in Eq.(13) by using shifted Chebyshev polynomials. As an example, the four-term asymptotic series in Eq.(12) is economized to a two-term expression as follows. When $\epsilon^{1/3}$ is factored out, Eq.(12) becomes

$$x_4(\epsilon) \sim \epsilon^{1/3}\left(3^{1/3} - \frac{2}{5}\epsilon^{2/3} + \frac{9}{175}3^{2/3}\epsilon^{4/3} - \frac{2}{175}3^{1/3}\epsilon^{6/3}\right). \tag{19}$$

Let $t = \epsilon^{2/3}$, then, Eq.(19) becomes

$$x_4(t) \sim \sqrt{t}\left(3^{1/3} - \frac{2}{5}t + \frac{9}{175}3^{2/3}t^2 - \frac{2}{175}3^{1/3}t^3\right). \tag{20}$$

Define

$$P(t) = 3^{1/3} - \frac{2}{5}t + \frac{9}{175}3^{2/3}t^2 - \frac{2}{175}3^{1/3}t^3, \tag{21}$$

we can use Chebyshev polynomials to economize the power series $P(t)$ (Abramowitz and Stegun, 1970; Cheney, 1982). Since the particular interest to engineering applications is in the range of $\phi \in (0, 45°)$, corresponding to $\epsilon \in (0, 0.215)$ and $t = \epsilon^{2/3} \in (0, 1/3)$, we use shifted Chebyshev polynomials to get the final result as follows.

$$\begin{aligned} P(t) = {} & \left(3^{1/3} - \frac{3^{2/3}}{1400} + \frac{3^{1/3}}{15120}\right) - \left(\frac{2}{5} - \frac{3 * 3^{2/3}}{175} + \frac{3^{1/3}}{840}\right)t \\ & + \left(\frac{3^{2/3}}{1400} - \frac{3^{1/3}}{12600}\right)T_2 + \frac{3^{1/3}}{75600}T_3 \end{aligned} \tag{22}$$

Where T_i's are the terms of Chebyshev polynomials with $\|T_i\| < 1$

Let
$$P_c(t) = \left(3^{1/3} - \frac{3^{2/3}}{1400} + \frac{3^{1/3}}{15120}\right) - \left(\frac{2}{5} - \frac{3 * 3^{2/3}}{175} + \frac{3^{1/3}}{840}\right)t \tag{23}$$

Then
$$\begin{aligned} \|P(t) - P_c(t)\| & \le \left\|\left(\frac{3^{2/3}}{1400} - \frac{3^{1/3}}{12600}\right)T_2\right\| + \left\|\frac{3^{1/3}}{75600}T_3\right\| \\ & \le \left(\frac{3^{2/3}}{1400} - \frac{3^{1/3}}{12600}\right) + \frac{3^{1/3}}{75600} \\ & < 1.390387 * 10^{-3} \end{aligned} \tag{24}$$

Therefore, the following economized asymptotic series solution can be obtained

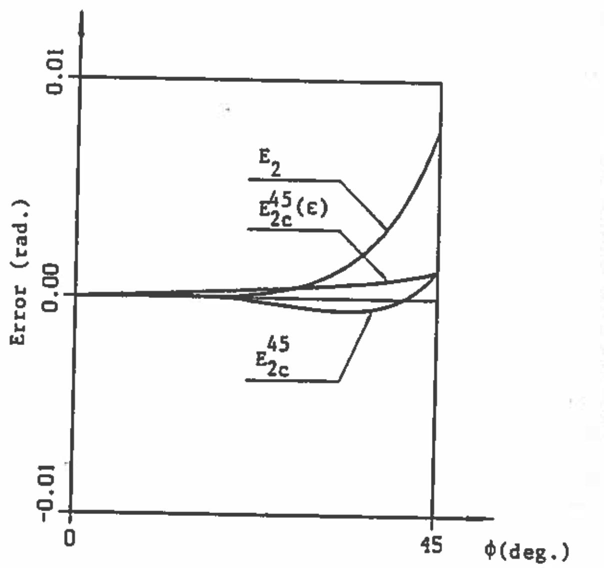

Figure 3 Comparison of the estimated and exact errors of the economized asymptotic solution x_{2c} for $0 < \phi < 45°$ with the exact error E_2 of the simple expansion $x_2(\epsilon)$.

$$\begin{aligned} x_{2c}^{45}(\epsilon) & \sim \epsilon^{1/3}P_c(t) \sim \epsilon^{1/3}P_c(\epsilon^{2/3}) \\ & \sim \left(3^{1/3} - \frac{3^{2/3}}{1400} + \frac{3^{1/3}}{15120}\right)\epsilon^{1/3} - \left(\frac{2}{5} - \frac{3 * 3^{2/3}}{175} + \frac{3^{1/3}}{840}\right)\epsilon \\ & \sim 1.440859\epsilon^{1/3} - 0.3660584\epsilon. \end{aligned} \tag{25}$$

The expression for the estimated error can be derived as follows

$$\begin{aligned} E_{2c}^{45}(\epsilon) & = \|x - x_{2c}^{45}(\epsilon)\| \\ & \le \|x - x_4(\epsilon)\| + \|x_4(\epsilon) - x_{2c}^{45}(\epsilon)\| \\ & \sim \frac{2 * 3^{1/3}}{175}\epsilon^{7/3} + \left(\frac{3^{2/3}}{1400} - \frac{3^{1/3}}{15120}\right)\epsilon^{1/3} \\ & \sim 1.390387 * 10^{-3}\epsilon^{1/3} + 0.0164828\epsilon^{7/3} \end{aligned} \tag{26}$$

If we only consider the range of $\phi \in (0, 35°)$ for $\epsilon \in (0, 0.1)$ and $t = \epsilon^{2/3} \in (0, 0.2)$, following the same derivation as shown above, we can get the approximate solution and the corresponding expression for the estimated error as follows.

$$\begin{cases} x_{2c}^{35}(\varepsilon) \sim 1.441735\varepsilon^{1/3} - 0.379223\varepsilon \\ E_{2c}^{35}(\varepsilon) \sim 0.514274 * 10^{-3}\varepsilon^{1/3} + 0.0164828\varepsilon^{7/3} \end{cases} \tag{27}$$

In Table 2, we compare the economized asymptotic solutions $x_{2c}^{35}(\varepsilon)$, $x_{2c}^{45}(\varepsilon)$ and the estimated errors $E_{2c}^{35}(\varepsilon), E_{2c}^{45}(\varepsilon)$ with their exact values. From Table 2, we can see that when ϕ is within the interval $(0, 35°)$, $x_{2c}^{35}(\varepsilon)$ is more accurate than $x_{2c}^{45}(\varepsilon)$. Both the exact error E_{2c}^{45} and estimated error $E_{2c}^{45}(\varepsilon)$ are compared with the exact error E_2 of the simple expansion $x_2(\varepsilon) = (3\varepsilon)^{1/3} - 2\varepsilon/5$ in Fig.3, where the error is defined as

$$E_{2c}^{45} = x_{exact} - x_{2c}^{45}(\varepsilon) \tag{28}$$

The improvement of the accuracy is obvious.

3. Approximation of $\text{inv}^{-1}(\varepsilon)$ when $|\varepsilon| \gg 1$

When $\varepsilon = \text{inv}\phi$ becomes large, the asymptotic approximate solution (16) or (17) will fail, and we have to seek another expression for the solution of $\text{inv}^{-1}(\varepsilon)$. According to Fig. 2, when ε approaches infinity, x will approach $\pi/2$. Let

$$\eta = 1/\varepsilon = 1/\text{inv}\phi, \quad \phi = x, \quad \eta \ll 1, \tag{29}$$

$$x = \pi/2 - y, \quad y = \pi/2 - x, \quad y \ll 1, \tag{30}$$

then Eq.(4) becomes

$$\tan(\frac{\pi}{2} - y) - (\frac{\pi}{2} - y) = \frac{1}{\eta}, \tag{31}$$

$$\eta(\cot y + y - \frac{\pi}{2}) = 1. \tag{32}$$

Expanding $\cot y$ in the Taylor series for $0 < |y| < \pi$, we get

$$\cot y = \frac{1}{y} - \frac{1}{3}y - \frac{1}{45}y^3 - \frac{2}{945}y^5 - \cdots - \frac{2^{2n} B_n y^{2n-1}}{(2n)!} - \cdots. \tag{33}$$

Substituting Eq.(33) into Eq.(32), we get

$$\eta\Big[(\frac{1}{y} - \frac{1}{3}y - \frac{1}{45}y^3 - \frac{2}{945}y^5 - \cdots) + y - \frac{\pi}{2}\Big] = 1. \tag{34}$$

Let the asymptotic solution of Eq.(34) be the perturbation series

$$y = \sum_{n=1}^{\infty} \delta_n(\eta) y_n, \quad \text{as} \quad \eta \to 0^+, \tag{35}$$

where $\delta_n(\eta) \ll \text{ord}(1)$ and $\delta_n(\eta) y_n \gg \delta_{n+1}(\eta) y_{n+1}$, as $\eta \to 0^+$.

The details of the derivation for δ_i and y_i are given in Appendix A. The final results for $x = \text{inv}^{-1}(\varepsilon)$ and error $E(\varepsilon)$ are as follows:

$$\begin{cases} x_6(\varepsilon) \sim \frac{\pi}{2} - \frac{1}{\varepsilon} + \frac{\pi}{2\varepsilon^2} - [(\frac{\pi}{2})^2 + \frac{2}{3}]\frac{1}{\varepsilon^3} + [(\frac{\pi}{2})^3 + \pi]\frac{1}{\varepsilon^4} \\ \qquad - [(\frac{\pi}{2})^4 + \pi^2 + \frac{13}{15}]\frac{1}{\varepsilon^5} + \cdots, \\ E_6(\varepsilon) \sim (\frac{\pi^4}{16} + \pi^2 + \frac{13}{15})\frac{1}{\varepsilon^5}. \end{cases} \tag{36}$$

In order to satisfy the condition of asymptoticity: $\delta_n y_n \gg \delta_{n+1} y_{n+1}$, as $\eta \to 0^+$, i.e., $\varepsilon \to +\infty$. Let

$$\eta^n y_n \gg \eta^{n+1} y_{n+1} \quad \text{or} \quad \varepsilon = \frac{1}{\eta} \gg \frac{y_{n+1}}{y_n}. \tag{37}$$

Writing expression (37) explicitly, we get

$$\varepsilon \gg \frac{y_2}{y_1} = \frac{\pi}{2},$$

$$\varepsilon \gg \frac{y_3}{y_2} = \frac{\pi^2/4 + 2/3}{\pi/2} \approx 2,$$

$$\varepsilon \gg \frac{y_4}{y_3} = \frac{\pi^3/8 + \pi}{\pi^2/4 + 2/3} \approx 2.24,$$

$$\text{and} \quad \varepsilon \gg \frac{y_5}{y_4} \approx 2.4$$

The computer results in Table 3 show that when $\varepsilon = 1/\eta \gg 5$, the asymptotic solution (36) is a good approximation of the exact solution of $\text{inv}^{-1}(\varepsilon)$. The bigger the value of ε, the more accurate is the asymptotic solution. But when $\phi \in (72°, 81°)$, both asymptotic formulae (17) and (36) derived so far are not good approximations. Therefore, we need to develop another asymptotic solution for $\phi = \text{inv}^{-1}(\varepsilon)$ which will be accurate for the intermediate values of ε.

4. Approximation of $\text{inv}^{-1}(\varepsilon)$ when $|\varepsilon| \sim 1$

We seek asymptotic solution when ϕ is about 75° or $5\pi/12$ and $\text{inv}(5\pi/12) \approx 2.42$. Let

$$\zeta = \text{inv}(\phi) - \text{inv}(5\pi/12), \quad x = z + 5\pi/12, \quad \zeta, z \ll 1. \tag{38}$$

Substituting Eq.(38) into Eq.(4), we get

$$\frac{\tan(5\pi/12) + \tan z}{1 - \tan(5\pi/12)\tan z} - z - \tan(5\pi/12) = \zeta. \tag{39}$$

Since $\text{inv}(\phi) \approx 2, \tan(5\pi/1)\tan z < \text{ord}(1)$, and Eq.(39) becomes

$$(\tan\frac{5\pi}{12} + \tan z)\sum_{n=1}^{\infty}(\tan\frac{5\pi}{12}\tan z)^n - z - \tan\frac{5\pi}{12} = \zeta. \tag{40}$$

Table 2: Comparison of $x_{2c}(\varepsilon) = \text{inv}^{-1}(\varepsilon)$ and estimated errors $E_{2c}(\varepsilon)$ after series economization with their exact values for $|\varepsilon| \ll 1$

$\varepsilon = \text{inv}(x_{exact})$	x_{exact}	x_{2c}^{45}	E_{2c}^{45} (radian)		x_{2c}^{35}	E_{2c}^{35} (radian)	
			E_{exact}	$E_{estimate}$		E_{exact}	$E_{estimate}$
$1.77 * 10^{-6}$	1°	59′56.54″	$1.68 * 10^{-5}$	$1.68 * 10^{-5}$	59′58.72″	$6.19 * 10^{-6}$	$6.22 * 10^{-6}$
$2.22 * 10^{-4}$	5°	4°59′44.17″	$7.68 * 10^{-5}$	$8.42 * 10^{-5}$	4°59′54.51″	$2.66 * 10^{-5}$	$3.11 * 10^{-5}$
$1.79 * 10^{-3}$	10°	9°59′37.13″	$1.11 * 10^{-4}$	$1.69 * 10^{-4}$	9°59′54.22″	$2.80 * 10^{-5}$	$6.25 * 10^{-5}$
$6.14 * 10^{-3}$	15°	14°59′45.98″	$6.80 * 10^{-5}$	$2.55 * 10^{-4}$	15°0′2.39″	$-1.16 * 10^{-5}$	$9.43 * 10^{-5}$
$1.47 * 10^{-2}$	20°	20°14.04″	$-6.81 * 10^{-5}$	$3.43 * 10^{-4}$	20°0′18.04″	$-8.75 * 10^{-5}$	$1.27 * 10^{-4}$
$3.00 * 10^{-2}$	25°	25°0′57.90″	$-2.81 * 10^{-4}$	$4.37 * 10^{-4}$	25°0′32.65″	$-1.58 * 10^{-4}$	$1.64 * 10^{-4}$
$5.36 * 10^{-2}$	30°	30°1′42.92″	$-4.99 * 10^{-4}$	$5.43 * 10^{-4}$	30°0′25.16″	$-1.22 * 10^{-4}$	$2.12 * 10^{-4}$
$8.93 * 10^{-2}$	35°	35°1′55.68″	$-5.61 * 10^{-4}$	$6.80 * 10^{-4}$	34°59′13.87″	$2.24 * 10^{-4}$	$2.89 * 10^{-4}$
$1.41 * 10^{-1}$	40°	40°0′31.22″	$-1.51 * 10^{-4}$	$8.94 * 10^{-4}$			
$2.15 * 10^{-1}$	45°	44°55′30.03″	$1.31 * 10^{-3}$	$1.29 * 10^{-3}$			

Table 3: Comparison of $x(\varepsilon) = \mathrm{inv}^{-1}(\varepsilon)$ and estimated errors $E(\varepsilon)$ with their exact values for $|\varepsilon| \gg 1$

$\varepsilon = \mathrm{inv}(x_{exact})$	x_{exact}	$x_6(\varepsilon)$	E_6 (radian)	
			E_{exact}	$E_{estimate}$
4.90	81°	80°53′ 7.23″	$2.00 * 10^{-3}$	$5.96 * 10^{-3}$
6.70	83°	82°58′50.14″	$3.39 * 10^{-4}$	$1.25 * 10^{-3}$
9.95	85°	84°59′52.79″	$3.49 * 10^{-5}$	$1.73 * 10^{-4}$
17.60	87°	86°59′59.68″	$1.57 * 10^{-6}$	$1.01 * 10^{-5}$
55.70	89°	88°59′59.93″	$3.15 * 10^{-7}$	$3.13 * 10^{-6}$

Table 4:Comparison of $x(\varepsilon) = \mathrm{inv}^{-1}(\varepsilon)$ and estimated errors $E(\varepsilon)$ with their exact values for $|\varepsilon| \sim 1$

$\varepsilon = \mathrm{inv}(x_{exact})$	x_{exact}	$x_5(\varepsilon)$	E_5 (radian)	
			E_{exact}	$E_{estimate}$
1.82	72°	72°0′16.56″	$8.03 * 10^{-5}$	$1.63 * 10^{-4}$
2.20	74°	73°59′59.61″	$1.89 * 10^{-6}$	$3.30 * 10^{-6}$
2.68	76°	76°0′1.69″	$8.17 * 10^{-6}$	$5.77 * 10^{-6}$
3.34	78°	78°1′30.46″	$4.39 * 10^{-4}$	$8.88 * 10^{-4}$
4.90	81°	80°56′40.28″	$9.68 * 10^{-4}$	$4.66 * 10^{-2}$

Using the identity $\tan(5\pi/12) = 2 + \sqrt{3}$, Eq.(40) becomes

$$(8 + 4\sqrt{3}) \sum_{n=1}^{\infty} (2 + \sqrt{3})^{n-1} (\tan z)^n - z = \zeta. \tag{41}$$

Using the Taylor series for $\tan z$ and perturbation series for $z(\zeta)$, finally, we obtain (see Appendix B) the asymptotic solution

$$\begin{aligned} z_5(\zeta) \sim\ & \frac{5\pi}{12} + (7 - 4\sqrt{3})\zeta - (388 - 224\sqrt{3})\zeta^2 \\ & + \frac{1}{3}(323565\sqrt{3} - 560431)\zeta^3 \\ & - \frac{1}{3}(97383044\sqrt{3} - 168672380)\zeta^4 + \cdots, \end{aligned} \tag{42}$$

and the estimated error

$$E_5(\zeta) \sim \frac{1}{3}(97383044\sqrt{3} - 168672380)\zeta^4, \tag{43}$$

where $\zeta = \varepsilon - \mathrm{inv}(5\pi/12)$. In Table 4, the exact and asymptotic solutions, and the exact and estimated errors are listed.

5. Composite Approximate Solution of $\phi = \mathrm{inv}^{-1}(\varepsilon)$ for $-\infty < \varepsilon < +\infty$

Combining the above three cases together, we get the following composite formulae for the inverse involute function $\phi = \mathrm{inv}^{-1}(\varepsilon)$ in Eq.(1) as follows.

$$\phi \sim \begin{cases} \phi_1 = 3^{1/3}\varepsilon^{1/3} - \frac{2}{5}\varepsilon + \frac{9}{175}3^{2/3}\varepsilon^{5/3} - \frac{2}{175}3^{1/3}\varepsilon^{7/3} \\ \quad - \frac{144}{67375}\varepsilon^3 + \frac{3258}{3128125}3^{2/3}\varepsilon^{11/3} \\ \quad - \frac{49711}{153278125}3^{1/3}\varepsilon^{13/3} - \cdots, & |\varepsilon| < 1.8 \\ \phi_2 = \frac{5\pi}{12} + (7 - 4\sqrt{3})\zeta - (388 - 224\sqrt{3})\zeta^2 \\ \quad + \frac{1}{3}(323565\sqrt{3} - 560431)\zeta^3 \\ \quad - \frac{1}{3}(97383044\sqrt{3} - 168672380)\zeta^4 + \cdots, & 1.8 < |\varepsilon| < 5 \\ \phi_3 = \frac{\pi}{2} - \frac{1}{\varepsilon} + \frac{\pi}{2\varepsilon^2} - [(\frac{\pi}{2})^2 + \frac{2}{3}]\frac{1}{\varepsilon^3} + [(\frac{\pi}{2})^3 + \pi]\frac{1}{\varepsilon^4} \\ \quad - [(\frac{\pi}{2})^4 + \pi^2 + \frac{13}{15}]\frac{1}{\varepsilon^5} + \cdots, & |\varepsilon| > 5 \end{cases} \tag{44}$$

where $\zeta = \varepsilon - \mathrm{inv}(5\pi/12)$, or

$$\phi \sim \begin{cases} \phi_1 = 1.44225\varepsilon^{1/3} - 0.4\varepsilon + 0.106976\varepsilon^{5/3} - 0.0164828\varepsilon^{7/3} \\ \quad - 0.213729 * 10^{-2}\varepsilon^3 + 0.216645 * 10^{-2}\varepsilon^{11/3} \\ \quad - 0.467749 * 10^{-3}\varepsilon^{13/3} + \cdots, & |\varepsilon| < 1.8 \\ \phi_2 = 1.308997 + 0.0717968\zeta - 0.02061910\zeta^2 \\ \quad + 0.6517008 * 10^{-2}\zeta^3 \\ \quad - 0.1238501 * 10^{-2}\zeta^4 + \cdots, & 1.8 < |\varepsilon| < 5 \\ \phi_3 = \frac{\pi}{2} - \frac{1}{\varepsilon} + \frac{\pi}{2\varepsilon^2} - 3.134068\frac{1}{\varepsilon^3} + 7.017377\frac{1}{\varepsilon^4} \\ \quad - 16.82434\frac{1}{\varepsilon^5} + \cdots, & |\varepsilon| > 5 \end{cases} \tag{45}$$

The corresponding estimated error expressions are

$$E(\varepsilon) \sim \begin{cases} \frac{1130112}{9306171875}\varepsilon^5, & |\varepsilon| < 1.8 \\ \frac{1}{3}(97383044\sqrt{3} - 168672380)(\varepsilon - \mathrm{inv}(5\pi/12))^4, & 1.8 < |\varepsilon| < 5 \\ (\frac{\pi^4}{16} + \pi^2 + \frac{13}{15})\frac{1}{\varepsilon^5}, & |\varepsilon| > 5 \end{cases} \tag{46}$$

or

$$E(\varepsilon) \sim \begin{cases} 1.21437 * 10^{-4}\varepsilon^5, & |\varepsilon| < 1.8 \\ 1.238501 * 10^{-3}(\varepsilon - 2.423054)^4, & 1.8 < |\varepsilon| < 5 \\ 16.82434\frac{1}{\varepsilon^5}, & |\varepsilon| > 5 \end{cases} \tag{47}$$

It should be mentioned that when $\varepsilon < 1$, corresponding to $\phi < 65°$, the accuracy of the nine term expression of $x(\varepsilon)$ in Eq.(17) is better than that of the seven term expression in Eq.(44) for $|\varepsilon| < 1$; when $1 < \varepsilon < 1.8$, the performance of these two expressions is just exchanged.

6. Numerical Example

The following application example illustrates the accuracy and usefulness of the above explicit formulae of the inverse involute function.

Problem Two spur gears of 12 and 15 teeth, respectively, are to be cut by a 20° full-depth 6-pitch hob. Determine the center distance at which to generate the gears to avoid undercutting (Mabie and Reinholtz, 1987, p.183).

$$e_1 = \frac{1}{P_d}(k - \frac{N_1}{2}\sin^2\phi) = 0.04968 \text{ in.}$$

$$e_2 = \frac{1}{P_d}(k - \frac{N_2}{2}\sin^2\phi) = 0.02045 \text{ in.}$$

$$\text{inv}\phi' = \text{inv}\phi + \frac{2P_d(e_1+e_2)\tan\phi}{N_1+N_2} = 0.02624$$

Using Eq.(16), we get

$$\begin{aligned}\phi' &= \text{inv}^{-1}(0.02624)\\ &= (3 * 0.02624)^{\frac{1}{3}} - \frac{2}{5} * 0.02624 \qquad \text{(rad.)}\\ &= 23.95424862^\circ.\end{aligned}$$

$$r_1' = \frac{r_1\cos\phi}{\cos\phi'} = \frac{1 * \cos 20^\circ}{\cos\phi'} = 1.028256395 \text{ in.}$$

$$r_2' = \frac{r_2\cos\phi}{\cos\phi'} = \frac{1.25 * \cos 20^\circ}{\cos\phi'} = 1.285320494 \text{ in.}$$

$$C' = r_1' + r_2' = 2.313576889 \text{ in.}$$

with 0.0108634% error in C'. The exact values obtained by numerically solving the non-linear equation (4) are $\phi' = 23.9682549^\circ$, and $C' = 2.313828251$ in., which match the results calculated by using Eq.(17) or the first expression of Eq.(44). The values obtained by using the four term expression of Eq.(12) are $\phi' = 23.9682573^\circ$, and $C' = 2.313828294$ in. (with $1.8 * 10^{-6}$% error). The values in (Mabie and Reinholtz, 1987, p.183), obtained by using linear interpolation from a table, are $\phi' = 23.97^\circ$, and $C' = 2.3144$ in. (with 0.02471% error). Therefore, the accuracy using the two term expression (16) is comparable to that using the linear interpolation method.

7. Conclusions

The explicit series solutions and estimated error expressions for the inverse involute function $\phi = \text{inv}^{-1}(\varepsilon)$ are derived by using perturbation techniques. When $|\varepsilon| < 0.215$, i.e., $|\phi| < 45^\circ$, which is the usual range for the tooth geometry calculations of involute gears, involute splines, and involute serrations, we can just use the two term expression $x_2(\varepsilon)$ of Eq.(16) with error in ϕ less than 1.0% of the range of ϕ or $7.76 * 10^{-3}$ radian.

In order to improve the accuracy of the two term approximation of the exact solution, we can use the economized asymptotic solution $x_{2c}^{45}(\varepsilon)$ in Eq.(25), with error for ϕ less than 0.17% or $1.31 * 10^{-3}$ radian. Sometimes, we only consider the range of $\phi \in (0, 35^\circ)$ for most applications; then, we can use economized asymptotic solution $x_{2c}^{35}(\varepsilon)$, which has error less than 0.04% or $2.24 * 10^{-4}$ radian for $\phi \in (0, 35^\circ)$.

If higher accuracy is required, we can use the first four, seven, or nine terms of Eq.(17), which have error less than 0.0018%, $4.89*10^{-6}$%, and $2.01*10^{-9}$% or $1.41*10^{-5}$, $3.84*10^{-8}$, and $1.58*10^{-9}$ radians for $\phi \in (0, 45^\circ)$, respectively; the maximum error occurs at $\phi = 45^\circ$. This accuracy should be satisfactory for practical applications. Under any circumstance, if the explicit solution from Eq.(44) or (45) is used as the initial value for the Newton's iteration algorithm, the convergence of the algorithm is guaranteed and the perfect value of ϕ (reach the precision limit of the computer) can be obtained in less three iteration steps.

The accuracy using the two term inverse involute series Eq.(16) is almost the same as the accuracy using the linear interpolation technique from a table of (ϕ,invϕ). Therefore, the use of the explicit formulae derived in this paper, instead of an extensive table of (ϕ,invϕ), are suggested in the tooth geometry calculations of the involute gears, splines, and serrations. Since the asymptotic series renders the implicit inverse involute function explicit, it is expected that these explicit formulae will find their applications not only in the tooth geometry calculations of gears, splines, and serrations, but also in some other fields of engineering where the inverse involute function is involved.

It is hoped that the method delineated in this paper, which incorporates the perturbation techniques into the interval method, will find its applications in solving other nonlinear problems.

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Appendix A. Derivation Details for $|\varepsilon| \gg 1$

Eqs.(34) and (35) are repeated here for convenience.

$$\eta\left[\left(\frac{1}{y} - \frac{1}{3}y - \frac{1}{45}y^3 - \frac{2}{945}y^5 - \cdots\right) + y - \frac{\pi}{2}\right] = 1, \qquad (A-1)$$

$$y \sim \sum_{n=1}^{\infty} \delta_n(\eta) y_n, \quad \delta_n(\eta) \ll \delta_{n-1}(\eta), \quad \text{as} \quad \eta \to 0^+. \quad (A-2)$$

To find the leading order term of y, let

$$y \sim \delta_1 y_1. \qquad (A-3)$$

Substituting expression(A-3) into Eq.(A-1), we get

$$\eta\left[\frac{1}{\delta_1 y_1} + \text{ord}(1)\right] \sim 1.$$

Let ord(1)=ord(η/δ_1) or $\delta_1 = \eta$; then, $1/y_1 = 1$ or $y_1 = 1$. The leading order approximation of y becomes

$$y \sim \eta. \qquad (A-4)$$

Substituting the following expression into Eq.(A–1),

$$y \sim \eta + \eta^2 y_2 + \eta^3 y_3 + \eta^4 y_4 + \eta^5 y_5 + \cdots. \quad (A-5)$$

we get

$$\begin{aligned} &\frac{1}{1+\eta y_2+\eta^2 y_3+\eta^3 y_4+\eta^4 y_5+o(\eta^5)} \\ &+\frac{2}{3}\eta[\eta+\eta^2 y_2+\eta^3 y_3+0(\eta^4)] \\ &-\frac{1}{45}\eta[\eta+\eta^2 y_2+0(\eta^3)]-\frac{\pi}{2}\eta \sim 1. \end{aligned} \quad (A-6)$$

$$\begin{aligned} &[1-(1+\eta y_2+\eta^2 y_3+\eta^3 y_4+\eta^4 y_5) \\ &\quad +(1+\eta y_2+\eta^2 y_3+\eta^3 y_4+\eta^4 y_5)^2 \\ &\quad -(1+\eta y_2+\eta^2 y_3+\eta^3 y_4+\eta^4 y_5)^3 \\ &\quad +(1+\eta y_2+\eta^2 y_3+\eta^3 y_4+\eta^4 y_5)^4-0(\eta^5)] \\ &\quad +\frac{2}{3}[\eta^2+\eta^3 y_2+\eta^4 y_3+0(\eta)^5]-[\frac{1}{45}\eta^4+0(\eta^5)]-\frac{\pi}{2}\eta \sim 1. \end{aligned} \quad (A-7)$$

The following equations can be obtained by using the dominant balance method

$$\begin{cases} 1=1, & \mathrm{ord}(1), \\ -y_2/2-\pi/2=0, & \mathrm{ord}(\eta), \\ -y_3+y_2^2+2/3=0, & \mathrm{ord}(\eta^2), \\ -y_4+2y_2y_3-y_2^3+2y_2/3=0, & \mathrm{ord}(\eta^3), \\ -y_5+y_3^2+2y_2y_4-3y_2^2y_3+y_2^4+2y_3/3-1/45=0, & \mathrm{ord}(\eta^4). \end{cases} \quad (A-8)$$

After solving Eqs.(A–8), we finally obtain $y_2 = -\pi/2, y_3 = (\pi/2)^2 + 2/3, y_4 = -(\pi/2)^3 - \pi, y_5 = (\pi/2)^4 + \pi^2 + 13/15$. Hence

$$y \sim \eta-\frac{\pi}{2}\eta^2+(\frac{\pi^2}{4}+\frac{2}{3})\eta^3-(\frac{\pi^3}{8}+\pi)\eta^4+(\frac{\pi^4}{16}+\pi^2+\frac{13}{15})\eta^5+\cdots. \quad (A-9)$$

Substituting $\eta = 1/\varepsilon$ and $y = \pi/2 - x$ into expression(A-9), we will get the expression (36)

$$\begin{aligned} x(\varepsilon) \sim &\frac{\pi}{2}-\frac{1}{\varepsilon}+\frac{\pi}{2\varepsilon^2}-(\frac{\pi^2}{4}+\frac{2}{3})\frac{1}{\varepsilon^3} \\ &+(\frac{\pi^3}{8}+\pi)\frac{1}{\varepsilon^4}-(\frac{\pi^4}{16}+\pi^2+\frac{13}{15})\frac{1}{\varepsilon^5}+\cdots. \end{aligned} \quad (A-10)$$

Appendix B. Derivation Details for $|\varepsilon| \sim 1$

Eq.(4) becomes Eq.(41) with $\zeta = \mathrm{inv}\phi - \mathrm{inv}(5\pi/12)$ and $x = z + 5\pi/12$.

$$(8+4\sqrt{3})\sum_{n=1}^{\infty}(2+\sqrt{3})^{n-1}(\tan z)^n - z = \zeta, \quad (B-1)$$

$$\begin{aligned} &(8+4\sqrt{3})\tan z+(28+16\sqrt{3})(\tan z)^2+(104+53\sqrt{3})(\tan z)^3 \\ &\quad +(388+224\sqrt{3})(\tan z)^4+0(\tan^5 z)-z=\zeta. \end{aligned} \quad (B-2)$$

According to Taylor series of $\tan z$ in Eq.(5), Eq.(B-2) becomes

$$\begin{aligned} &(8+4\sqrt{3})(z+\frac{1}{3}z^3+\frac{2}{15}z^5+\cdots) \\ &\quad +(28+16\sqrt{3})(z+\frac{1}{3}z^3+\cdots)^2 \\ &\quad +(104+53\sqrt{3})(z+\frac{1}{3}z^3+\cdots)^3 \\ &\quad +(388+224\sqrt{3})(z+\frac{1}{3}z^3+\cdots)^4+0(z^5)-z=\zeta. \end{aligned} \quad (B-3)$$

Simplifying Eq.(B–3), we get

$$\begin{aligned} &(7+4\sqrt{3})z+(28+16\sqrt{3})z^2+\frac{320+163\sqrt{3}}{3}z^3 \\ &\quad +(388+224\sqrt{3})z^4+0(z^5)=\zeta. \end{aligned} \quad (B-4)$$

Let the solution to Eq.(B–4) be the asymptotic series

$$z(\zeta) \sim \sum_{n=1}^{\infty}\delta_n(\zeta)z_n, \qquad \delta_{n+1}(\zeta) \ll \delta_n(\zeta), \quad \text{as} \quad \zeta \to 0^+. \quad (B-5)$$

We can prove $\delta_n(\zeta) = \zeta^n$; therefore,

$$z(\zeta) \sim \zeta z_1 + \zeta^2 z_2 + \zeta^3 z_3 + \zeta^4 z_4 + \cdots. \quad (B-6)$$

For convenience, let $A_1 = (7+4\sqrt{3}), A_2 = (28+16\sqrt{3}), A_3 = (320+163\sqrt{3})/3, A_4 = (388+224\sqrt{3})$; then, Eq.(B–4) becomes

$$\begin{aligned} &A_1(\zeta z_1+\zeta^2 z_2+\zeta^3 z_3+\zeta^4 z_4+\cdots)+A_2(\zeta z_1+\zeta^2 z_2+\cdots)^2 \\ &\quad +A_3(\zeta z_1+\cdots)^3+A_4(\zeta z_1+\cdots)^4+0(\zeta^5)\sim\zeta. \end{aligned} \quad (B-7)$$

After expanding polynomial powers and using the dominant balance method, we obtain

$$\begin{cases} A_1 z_1 = 1, & \mathrm{ord}(\zeta), \\ A_1 z_2 + A_2 z_1^2 = 0, & \mathrm{ord}(\zeta^2), \\ A_1 z_3 + A_2 2 z_1 z_2 + A_3 z_1^3 = 0, & \mathrm{ord}(\zeta^3), \\ A_1 z_4 + A_2(2z_3^2 + 2z_1z_3) + A_3(3z_1^2 z_2) + A_4 z_1^4 = 0, & \mathrm{ord}(\zeta^4). \end{cases} \quad (B-8)$$

After solving the nonlinear system of Eqs.(B-8), we finally find

$$\begin{cases} z_1 = 7-4\sqrt{3} \approx 0.0717968 \\ z_2 = 224\sqrt{3}-388 \approx -0.02061910 \\ z_3 = (323565\sqrt{3}-560431)/3 \approx 0.6517008 * 10^{-2} \\ z_4 = (168672380-97383044\sqrt{3})/3 \approx -0.1238501 * 10^{-2} \end{cases}$$

Hence

$$\begin{aligned} z(\zeta) \sim &(7-4\sqrt{3})\zeta+(224\sqrt{3}-388)\zeta^2 \\ &+\frac{1}{3}(323565\sqrt{3}-560431)\zeta^3 \\ &+\frac{1}{3}(168672380-97383044\sqrt{3})\zeta^4+\cdots. \end{aligned} \quad (B-9)$$

Substituting $z = x - 5\pi/12$ into expression(B–9), we get

$$\begin{aligned} x(\zeta) \sim &\frac{5\pi}{12}+(7-4\sqrt{3})\zeta+(224\sqrt{3}-388)\zeta^2 \\ &+\frac{1}{3}(323565\sqrt{3}-560431)\zeta^3 \\ &+\frac{1}{3}(168672380-97383044\sqrt{3})\zeta^4+\cdots. \end{aligned} \quad (B-10)$$