EME172
Homework #2 Solution

2. Problem 3-4

Equations of motion in Laplace:

$$(s^2+3s+1)x_1(s)-(s+1)x_2(s)-sx_3(s)=0$$
$$-(s+1)x_1(s)+(s^2+2s+1)x_2(s)-sx_3(s)=F(s)$$
$$-sx_1(s)-sx_2(s)+(s^2+3s)x_3(s)=0$$

Equations of motion in the time domain:

$$\frac{d^2x_1}{dt^2}+3\frac{dx_1}{dt}+x_1-\frac{dx_2}{dt}-x_2-\frac{dx_3}{dt}=0$$
$$\frac{d^2x_2}{dt^2}+2\frac{dx_2}{dt}+x_2-\frac{dx_1}{dt}-x_1-\frac{dx_3}{dt}=f(t)$$
$$\frac{d^2x_3}{dt^2}+3\frac{dx_3}{dt}+x_1-\frac{dx_2}{dt}-\frac{dx_1}{dt}=0$$

Define state variables:

$$z_1=x_1 \quad \text{or} \quad x_1=z_1 \tag{1}$$
$$z_2=\frac{dx_1}{dt} \quad \text{or} \quad \frac{dx_1}{dt}=z_2 \tag{2}$$
$$z_3=x_2 \quad \text{or} \quad x_2=z_3 \tag{3}$$
$$z_4=\frac{dx_2}{dt} \quad \text{or} \quad \frac{dx_2}{dt}=z_4 \tag{4}$$
$$z_5=x_3 \quad \text{or} \quad x_3=z_5 \tag{5}$$
$$z_6=\frac{dx_3}{dt} \quad \text{or} \quad \frac{dx_3}{dt}=z_5 \tag{6}$$

Substituting Eq. (1) in (2), (3) in (4), and (5) in (6), we obtain, respectively:

$$\frac{dz_1}{dt} = z_2 \tag{7}$$

$$\frac{dz_3}{dt} = z_4 \tag{8}$$

$$\frac{dz_5}{dt} = z_6 \tag{9}$$

Substituting Eqs. (1) through (6) into the equations of motion in the time domain and solving for the derivatives of the state variables and using Eqs. (7) through (9) yields the state equations:

$$\frac{dz_1}{dt} = z_2$$

$$\frac{dz_2}{dt} = -z_1 - 3z_2 + z_3 + z_4 + z_6$$

$$\frac{dz_3}{dt} = z_4$$

$$\frac{dz_4}{dt} = z_1 + z_2 - z_3 - 2z_4 + z_6 + f(t)$$

$$\frac{dz_5}{dt} = z_6$$

$$\frac{dz_6}{dt} = z_2 + z_4 - 3z_6$$

The output is $x_3 = z_5$.

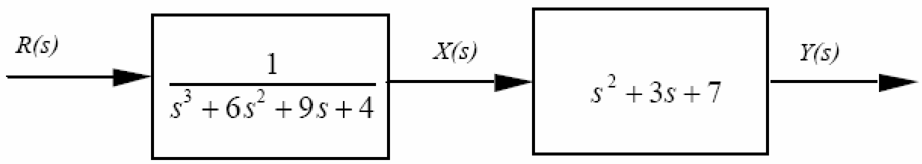
In vector-matrix form:

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -3 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -3 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} f(t)$$

$$y = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{z}$$

3. Problem 3-13

The transfer function can be represented as a block diagram as follows:



Writing the differential equation for the first box:

$$\dddot{x} + 6\ddot{x} + 9\dot{x} + 4x = r(t)$$

Defining the state variables:

$$x_1 = x$$
$$x_2 = \dot{x}$$
$$x_3 = \ddot{x}$$

Thus,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -4x - 9\dot{x} - 6\ddot{x} + r(t) = -4x_1 - 9x_2 - 6x_3 + r(t)$$

From the second box,

$$y = \ddot{x} + 3\dot{x} + 7x = 7x_1 + 3x_2 + x_3$$

In vector-matrix form:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -9 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$y = \begin{bmatrix} 7 & 3 & 1 \end{bmatrix}\mathbf{x}$$

4. Problem 3-14(b)

b. $G(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & -8 \\ 0 & 5 & 3 \\ -3 & -5 & -4 \end{pmatrix}; \mathbf{B} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}; \quad \mathbf{C} = (1, 3, 6)$$

$$(s\,\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^3 - 3s^2 - 27s + 157}\begin{pmatrix} s^2 - s - 5 & 3s + 52 & -8s + 49 \\ -9 & s^2 + 2s - 32 & 3s - 6 \\ -3s + 15 & -5s + 1 & s^2 - 7s + 10 \end{pmatrix}$$

Therefore, $G(s) = \dfrac{49s^2 - 349s + 452}{s^3 - 3s^2 - 27s + 157}$.

5. Problem 4-2(a)

a. $C(s) = \frac{5}{s(s+5)} = \frac{1}{s} - \frac{1}{s+5}$. Therefore, $c(t) = 1 - e^{-5t}$.

Also, $T = \frac{1}{5}$, $T_r = \frac{2.2}{a} = \frac{2.2}{5} = 0.44$, $T_s = \frac{4}{a} = \frac{4}{5} = 0.8$.

6. Problem 4-6

Writing the equation of motion,

$$(Ms^2 + 8s)X(s) = F(s)$$

Thus, the transfer function is,

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + 8s}$$

Differentiating to yield the transfer function in terms of velocity,

$$\frac{sX(s)}{F(s)} = \frac{1}{Ms + 8} = \frac{1/M}{s + \frac{8}{M}}$$

Thus, the settling time, T_s, and the rise time, T_r, are given by

$$T_s = \frac{4}{8/M} = \frac{1}{2}M; \quad T_r = \frac{2.2}{8/M} = 0.275M$$

7. Problem 4-8(b, d, e, f)

b. Poles: -3, -6; $c(t) = A + Be^{-3t} + Ce^{-6t}$; overdamped response.

d. Poles: $(-3+j3\sqrt{15})$, $(-3-j3\sqrt{15})$; $c(t) = A + Be^{-3t}\cos(3\sqrt{15}\ t + \phi)$; underdamped.

e. Poles: j3, -j3; Zero: -2; $c(t) = A + B\cos(3t + \phi)$; undamped.

f. Poles: -10, -10; Zero: -5; $c(t) = A + Be^{-10t} + Cte^{-10t}$; critically damped.